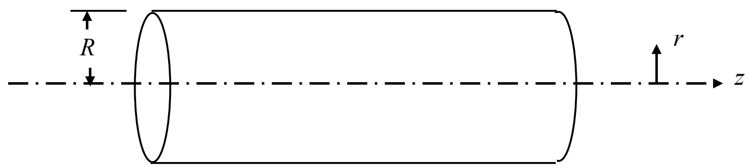
3. Poiseuille Flow

Consider the steady flow of an incompressible, constant viscosity, Newtonian fluid within an infinitely long, circular pipe of radius, R.



We'll make the following assumptions:

1. The flow is axi-symmetric and there is no "swirl" velocity. $\Rightarrow \frac{\partial}{\partial\theta}(\cdots) = 0$ and $u_\theta = 0$
2. The flow is steady. $\Rightarrow \frac{\partial}{\partial t}(\cdots) = 0$
3. The flow is fully-developed in the z-direction. $\Rightarrow \frac{\partial u_r}{\partial z} = \frac{\partial u_z}{\partial z} = 0$
4. There are no body forces. $\Rightarrow f_r = f_\theta = f_z = 0$

Let's first examine the continuity equation:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0$$

From assumptions #1 and #3 we see that:

$$ru_r = \text{constant}$$

Since there is no flow through the walls, the constant must be equal to zero and thus:

$$u_r = 0 \qquad \text{(call this condition \#5)}$$

Now let's examine the Navier-Stokes equation in the z-direction:

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho f_z$$

We can simplify this equation using our assumptions:

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0\ (\#2)} + \underbrace{u_r}_{=0\ (\#5)}\frac{\partial u_z}{\partial r} + \underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}}_{=0\ (\#1)} + u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0\ (\#3)}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial\theta^2}}_{=0\ (\#1)} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0\ (\#3)}\right] + \rho\underbrace{f_z}_{=0\ (\#4)}$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}$$

$$\Rightarrow r\frac{du_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + c_1$$

$$\Rightarrow u_z = \frac{r^2}{4\mu}\frac{dp}{dz} + c_1\ln r + c_2$$

Note that in the previous derivation the fact that u_z is a function only of r has been used to change the partial derivatives to ordinary derivatives. Furthermore, examining the Navier-Stokes equations in the r and θ directions demonstrates that the pressure, p, is a function only of z and thus ordinary derivatives can be used when differentiating the pressure with respect to z.

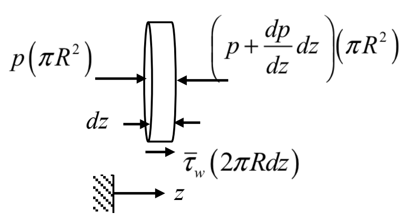
Now let's apply boundary conditions to determine the unknown constants c_1 and c_2. First, note that the fluid velocity in a pipe must remain finite as $r \rightarrow 0$ so that the constant c_1 must be zero (this is a type of kinematic boundary condition). Also, the pipe wall is fixed so that we have $u_z(r=R)=0$ (no-slip condition). After applying boundary conditions we have:

$$\boxed{u_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1-\frac{r^2}{R^2}\right)}$$ **Poiseuille Flow in a Circular Pipe**

Notes:

1. The velocity profile is a paraboloid with the maximum velocity occurring along the centerline. The average velocity in the pipe is found from:
$$\bar{u} = \frac{1}{\pi R^2}\int_{r=0}^{r=R} u_z\left(2\pi r dr\right) = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \tfrac{1}{2}u_{\max}$$
where $u_{\max}$ is the maximum fluid velocity.

2. As with planar Couette-Poiseuille flow, we can determine stresses using the constitutive relations for a Newtonian fluid. The shear stress that the pipe walls apply to the fluid, τ_w, is:
$$\tau_w = \frac{R}{2}\left(\frac{dp}{dz}\right) = \frac{-4\mu\bar{u}}{R}$$
where $\bar{u}$ is the average velocity in the pipe. Note that an alternate method for determining the average wall shear stress, which in this case is equal to the exact wall shear stress, is to balance shear forces and pressure forces on a small slice of the flow as shown below.



$$\sum F_z = 0 = p\pi R^2 - \left(p + \frac{dp}{dz}dz\right)\pi R^2 + \bar{\tau}_w 2\pi R dz$$

$$\bar{\tau}_w = \frac{R}{2}\frac{dp}{dz}$$ (The same answer as before!)

In engineering applications it is common to express the average shear stress in terms of a **(Darcy) friction factor, f_D,** which is defined as:
$$f_D \equiv \left|\frac{4\bar{\tau}_w}{\frac{1}{2}\rho\bar{u}^2}\right| = 64\left(\frac{\mu}{\rho\bar{u}D}\right) = \frac{64}{\text{Re}}$$
where $D=2R$ is the pipe diameter and Re is the Reynolds number. The Darcy friction factor commonly appears in the Moody chart for incompressible, viscous pipe flow. Note again that this solution is only valid only for a laminar flow. The condition for the flow to remain laminar is found experimentally to be:
$$\text{Re} \equiv \frac{\rho\bar{u}D}{\mu} < 2300$$

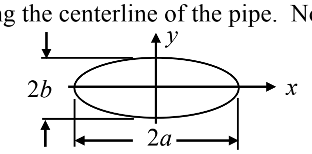
3. We can also use the general solution (before applying boundary conditions) to determine the flow between two concentric cylinders by applying different boundary conditions. For example, two fixed cylinders will have the boundary conditions: $u_z(r=R_I)=0$ and $u_z(r=R_O)=0$ where R_I and R_O are the inner and outer cylinder radii.

4. Laminar flow in an elliptical cross-section pipe can be determined by considering the simplified Navier-Stokes equation in the z-direction but using Cartesian coordinates (assuming u_x=u_y=0):

$$\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}=\frac{1}{\mu}\frac{dp}{dz} \qquad \text{(Poisson's equation!)}$$

where z is the coordinate along the centerline of the pipe. Note that the pipe wall boundary is the ellipse given by:

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$



where a and b are the lengths of the major and minor axes. Since we must satisfy the no-slip boundary condition at the pipe walls, let's guess that the solution has the form:

$$u_z=\alpha\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1\right]$$

since this profile automatically satisfies the boundary condition. The quantity α is an unknown constant. To determine if this is indeed a valid solution to the fluid equations, we first note that it satisfies the continuity equation (u_x=u_y=0 and u_z is not a function of z). If we substitute into the z-component of the Navier-Stokes equations (Poisson's equation above) we find that our guess for the velocity distribution is valid if the constant α is given by:

$$\alpha\left(\frac{2}{a^2}+\frac{2}{b^2}\right)=\frac{1}{\mu}\frac{dp}{dz}$$

$$\Rightarrow \quad \alpha=\frac{a^2b^2}{2\mu\left(a^2+b^2\right)}\frac{dp}{dz}$$

which means that the velocity profile for an elliptical pipe is given by:

$$\boxed{u_z=\frac{a^2b^2}{2\mu\left(a^2+b^2\right)}\frac{dp}{dz}\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1\right]}$$

velocity profile in a pipe of elliptical cross-section

For very complex cross-sections, we can determine the velocity profile by solving Poisson's equation numerically.