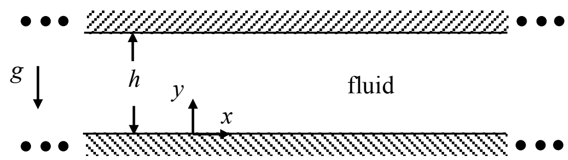
2. Planar Couette-Poiseuille Flow

Consider the steady flow of an incompressible, constant viscosity Newtonian fluid between two infinitely long, parallel plates separated by a distance, h.



We'll make the following assumptions:

1. The flow is planar. $\Rightarrow u_z = \text{constant}$ and $\frac{\partial}{\partial z}(\cdots) = 0$
2. The flow is steady. $\Rightarrow \frac{\partial}{\partial t}(\cdots) = 0$
3. The flow is fully-developed in the x-direction. $\Rightarrow \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = 0$
4. The only body force is that due to gravity in the $-y$-direction. $\Rightarrow f_x = 0$ and $f_y = -g$

Let's first examine the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

From assumption #3 we see that:

$$\frac{\partial u_x}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u_y}{\partial y} = 0$$

Based on this result and assumptions #1 and #3 we see that the y-velocity is a constant:

$$u_y = \text{constant}$$

Since there is no flow through the walls, the y-velocity must be zero.

$$u_y = 0 \qquad \text{(call this condition \#5)}$$

Now let's examine the Navier-Stokes equation in the y-direction:

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + \rho f_y$$

We can simplify this equation using our assumptions:

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0\ (\#2,\#5)} + u_x\underbrace{\frac{\partial u_y}{\partial x}}_{=0\ (\#3,\#5)} + \underbrace{u_y\frac{\partial u_y}{\partial y}}_{=0\ (\#5)}\right) = -\frac{\partial p}{\partial y} + \mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0\ (\#3,\#5)} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0\ (\#5)}\right) + \rho\underbrace{f_y}_{=-g\ (\#4)}$$

$$\Rightarrow \frac{\partial p}{\partial y} = -\rho g$$

$$\Rightarrow p(x,y) = f(x) - \rho g y \tag{1}$$

where $f(x)$ is an unknown function of x.

Now let's examine the Navier-Stokes equation in the x-direction:

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho f_x$$

After simplifying:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0\ (\#2)}+u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0\ (\#3)}+\underbrace{u_y}_{=0\ (\#5)}\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0\ (\#3)}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho\underbrace{f_x}_{=0\ (\#4)}$$

$$\Rightarrow\frac{\partial^2 u_x}{\partial y^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

Based on assumptions #1, #2, and #3 we can write:

$$\frac{\partial^2 u_x}{\partial y^2}=\frac{d^2 u_x}{dy^2}$$

so that the simplified Navier-Stokes equation in the x-direction becomes:

$$\frac{d^2 u_x}{dy^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice with respect to y (note that from Eq. (1) we observe that $\partial p/\partial x$ is not a function of y):

$$u_x=\frac{1}{2\mu}\frac{\partial p}{\partial x}y^2+c_1 y+c_2$$

where c_1 and c_2 are unknown constants that we find using our boundary conditions. Note that this is the equation of a parabola.

Let's examine the following case:

fixed bottom boundary: $u_x\left(y=0\right)=0$

top boundary moving with velocity, U: $u_x\left(y=h\right)=U$

After applying these boundary conditions to determine the constants c_1 and c_2 we find that the fluid velocity in the x-direction is given by:

$$\boxed{u_x=U\frac{y}{h}+\frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)}$$

This type of flow is often referred to as a **planar Couette-Poiseuille flow** (pronounced "'pwäz I")

Notes:

1. The stress acting on the fluid at any point can be found from the stress-strain rate constitutive relations for a Newtonian fluid.

$$\sigma_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

2. If we remove the pressure gradient and move the fluid using just the moving upper boundary, the velocity profile becomes linear:

$$u_x=U\frac{y}{h}$$

This type of flow is referred to as a **planar Couette flow**.

3. If we hold both boundaries stationary and move the fluid using only a pressure gradient (note that flow in the positive x-direction occurs for $dp/dx < 0$), the velocity profile becomes:

$$u_x = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)$$

   This type of flow is referred to as a **planar Poiseuille flow**.

4. The average flow velocity may be found by setting the volumetric flow rate using the average velocity equal to the volumetric flow rate using the real velocity profile. For example, for planar Poiseuille flow the average velocity is:

$$Q = \bar{u}h = \int_{y=0}^{y=h} \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)dy = \frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$\bar{u} = \frac{h^2}{12\mu}\left(-\frac{\partial p}{\partial x}\right) = \tfrac{2}{3}u_{\max} \tag{2}$$

5. Recall that we assumed that these solutions only hold for laminar flows (the u_y component is zero). Experimentally we observe that planar Couette-Poiseuille flow remains laminar for:

$$\mathrm{Re} = \frac{\rho \bar{u} h}{\mu} < 1500$$

   where Re is the Reynolds number of the flow and $\bar{u}$ is the average flow velocity. It should be noted that the value of 1500 is only approximate and can vary considerably depending on how carefully the experiment is conducted. Its value is given only as an engineering rule-of-thumb.

5. Velocity profiles for the various conditions are sketched below: