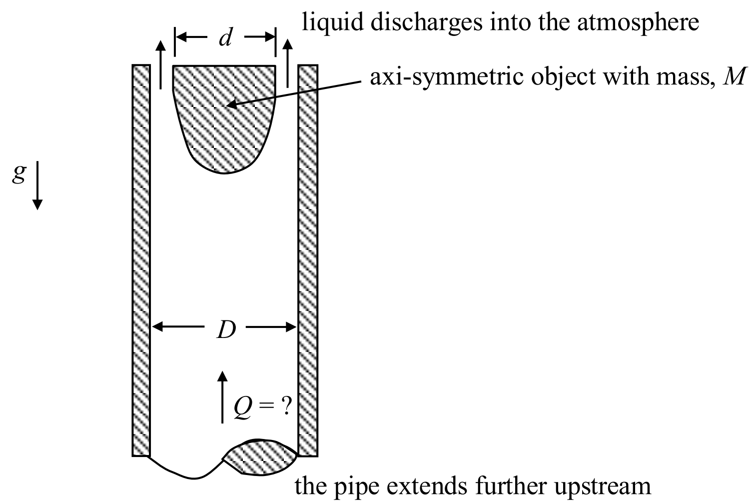
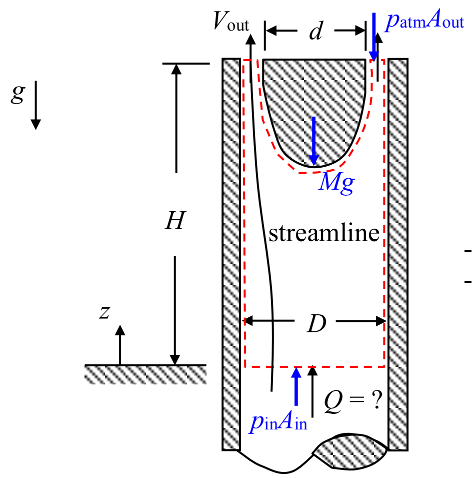
The axi-symmetric object shown below is placed in the end of a vertical circular pipe of inner diameter, D. A liquid with density, ρ, is pumped upward through the pipe and discharges to the atmosphere. Neglecting viscous effects, determine the volume flow rate, Q, of the liquid needed to support the object in the position shown in terms of d, D, g, ρ, and M.



SOLUTION:

Apply conservation of mass to the control volume shown below.



- Let: $A_{in} = \pi D^2/4$ and $A_{out} = \pi(D^2-d^2)/4$.
- Choose H such that it is much larger than the size of the object ($\Rightarrow V_{CV} \approx A_{in}H$).

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = -\rho Q + \rho V_{out} A_{out}$$

Substituting and simplifying gives:

$$-\rho Q + \rho V_{out} A_{out} = 0$$

$$V_{out} = \frac{Q}{A_{out}} \tag{2}$$

Apply the linear momentum equation in the z-direction to the same control volume. Use the fixed frame of reference shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV + \int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,z} + F_{S,z} \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\underbrace{m}_{=\rho Q}\left(\frac{Q}{A_{\text{in}}}\right) + mV_{\text{out}} \\ &= \rho Q\left(\frac{Q}{A_{\text{out}}} - \frac{Q}{A_{\text{in}}}\right) \\ &= \rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right)\end{aligned}$$

(Note that Eqn. (2) was used in simplifying the momentum flux term.)

$F_{B,z} = -\rho V_{\text{CV}} g \approx -\rho A_{\text{in}} H g$ (H is chosen to be much larger than the object size.)

$F_{S,z} = p_{\text{in}} A_{\text{in}} - Mg$ (use gage pressures so $p_{\text{out}} = p_{\text{atm}} = 0$)

Substitute and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right) = -\rho A_{\text{in}} H g + p_{\text{in}} A_{\text{in}} - Mg \tag{4}$$

To determine p_{in}, apply Bernoulli's equation along a streamline from the inlet to the outlet.

$$\left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{in}} = \left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{out}} \tag{5}$$

where

$p_{\text{in}} = ?$ $\qquad p_{\text{out}} = 0$ (gage pressure)

$V_{\text{in}} = \dfrac{Q}{A_{\text{in}}}$ $\qquad V_{\text{out}} = \dfrac{Q}{A_{\text{out}}}$ (from Eqn. (2))

$z_{\text{in}} = 0$ $\qquad z_{\text{out}} = H$

Substitute and simplify.

$$p_{\text{in}} = \tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2} - \frac{1}{A_{\text{in}}^2}\right) + \rho g H \tag{6}$$

Substitute Eqn. (6) into Eqn. (4) and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)=-\rho A_{\text{in}}Hg+\left[\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)+\rho gH\right]A_{\text{in}}-Mg$$

$$=\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)A_{\text{in}}-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}^2A_{\text{out}}^2}\right)A_{\text{in}}\right]=-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}A_{\text{out}}^2}\right)\right]=-Mg$$

$$\rho Q^2\left(\frac{2A_{\text{in}}A_{\text{out}}-2A_{\text{out}}^2-A_{\text{in}}^2+A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left(\frac{-A_{\text{in}}^2+2A_{\text{in}}A_{\text{out}}-A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left[\frac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}\right]=Mg$$

$$\boxed{\therefore Q=\sqrt{\frac{Mg}{\rho\dfrac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}}}=\frac{1}{\left(\dfrac{A_{\text{in}}}{A_{\text{out}}}-1\right)}\sqrt{\frac{2MgA_{\text{in}}}{\rho}}}\quad\text{where } A_{\text{in}}=\frac{\pi D^2}{4}\quad\text{and}\quad A_{\text{out}}=\frac{\pi\left(D^2-d^2\right)}{4}\qquad(7)$$