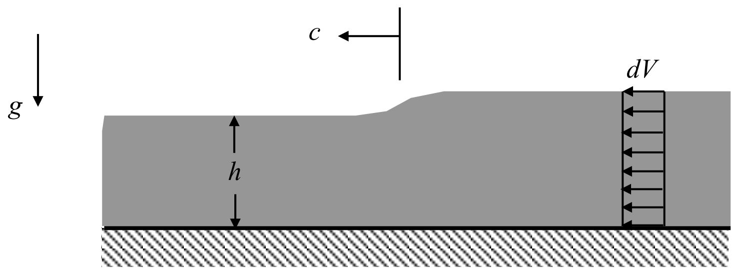
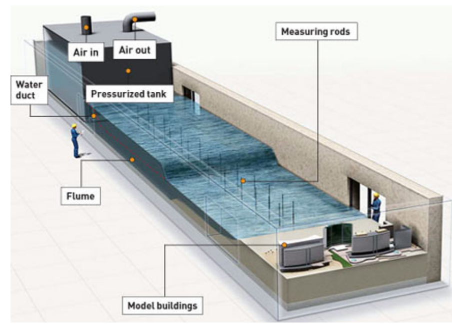
In an attempt to model the speed of a tsunami wave in the deep ocean, consider the propagation of a small amplitude, solitary wave front moving with speed, *c*, from right to left as shown in the figure below. Neglect the effects of surface tension. The liquid is initially at rest but after the wave passes by, the fluid behind the wave has a small velocity, *dV*, in the same direction as the wave.

Derive an expression for the wave speed, *c*. You may neglect the shear forces the channel bed and the atmosphere exert on the liquid. Hint: Consider choosing a steady frame of reference when analyzing the problem.



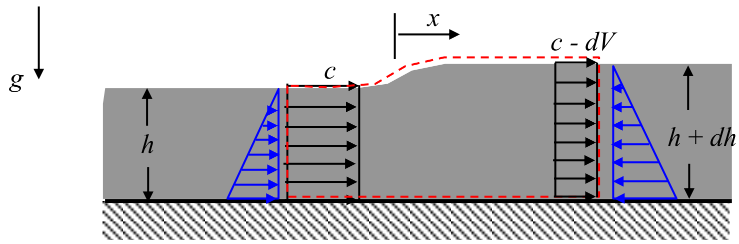


BRIEF SOLUTION:

1. Use a frame of reference fixed to the wave so that the flow appears steady.
2. Apply the linear momentum equation using a control volume that is perpendicular to the upstream and downstream flows where the velocities are uniform, along the bottom of the channel, and along the free surface. Be sure to include the pressure forces on the upstream and downstream boundaries. Note that the pressure increases linearly with depth from the free surface.
3. Apply conservation of mass to the same control volume.

DETAILED SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown below. Use a frame of reference that is fixed to the wave. Since a constant wave velocity is assumed, the frame of reference will be inertial.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CV} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CV} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = c(-\dot{m}) + (c - dV)\dot{m} = -\dot{m}dV \quad \text{where} \quad \dot{m} = \rho ch$$

$$F_{B,x} = 0$$

$$F_{S,x} = \tfrac{1}{2}\rho g h^2 - \tfrac{1}{2}\rho g (h + dh)^2 = -\rho g h dh - \underbrace{\tfrac{1}{2}\rho g dh^2}_{H.O.T.} = -\rho g h dh$$

Substitute and simplify.

$$-\rho ch dV = -\rho g h dh$$

$$c dV = g dh \qquad (1)$$

Apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CV} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CV} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho ch + \rho(c - dV)(h + dh) = -\rho ch + \rho ch + \rho c dh - \rho h dV - \underbrace{\rho dV dh}_{=H.O.T.}$$

$$= \rho c dh - \rho h dV$$

Substitute and simplify.

$$\rho c dh - \rho h dV = 0 \Rightarrow c dh = h dV$$

$$dh = \frac{h dV}{c} \qquad (2)$$

Substitute Eqn. (2) into Eqn. (1) and simplify.

$$c dV = g\frac{h dV}{c}$$

$$\boxed{\therefore c = \sqrt{gh}} \qquad (3)$$

As an example, consider the speed of a traveling wave in the deep ocean resulting from an undersea earthquake for example (the wave amplitude is small compared to its wavelength). Assuming an ocean depth of 1610 m (1 mile), the speed of the wave will be 126 m/s (280 mph)!