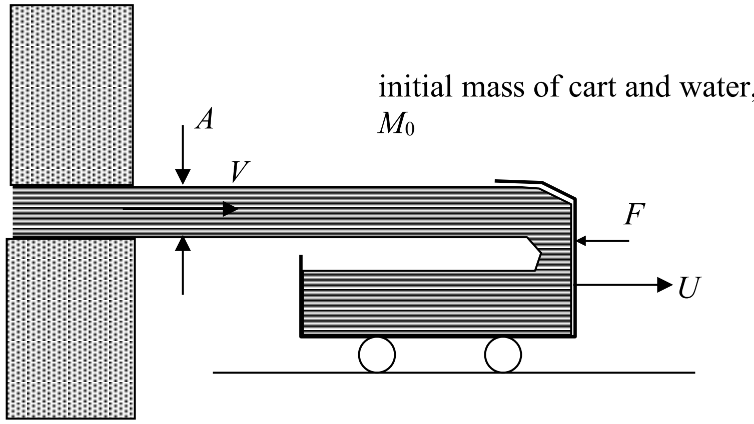
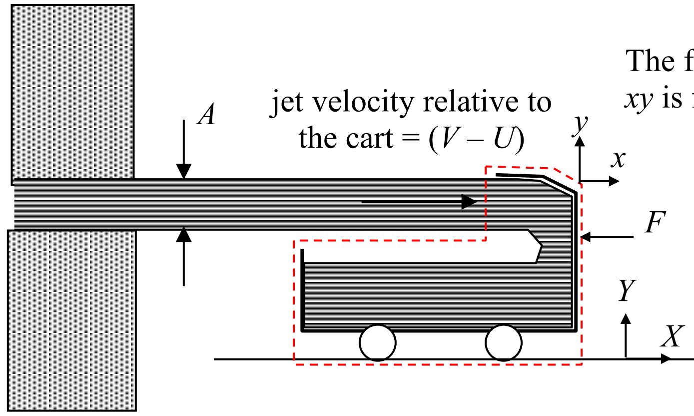
The tank shown rolls along a level track. Water received from a jet is retained in the tank. The tank is to accelerate from rest toward the right with constant acceleration, *a*. Neglect wind and rolling resistance. Find an algebraic expression for the force (as a function of time) required to maintain the tank acceleration at constant *a*.



SOLUTION:

First apply conservation of mass to a control volume surrounding the cart (shown below) in order to determine how the cart mass changes with time.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho\left(V-U\right)A$$

Substitute and re-arrange.

$$\frac{dM_{CV}}{dt} - \rho\left(V-U\right)A = 0$$

$$\frac{dM_{CV}}{dt} = \rho\left(V-U\right)A \tag{2}$$

Since the cart acceleration is constant (= a), we may write:

$U = at$ (Note that $U(t = 0) = 0$ since the cart starts from rest.) (3)

Note that Eqn. (3) is only true <u>when a = constant</u>. Otherwise, if $a = a(t)$ one must write the velocity as:

$$U = U_0 + \int_0^t a dt \quad (4)$$

Substitute Eqn. (3) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dM_{\text{CV}}}{dt} = \rho\left(V - at\right)A \quad (5)$$

$$\int_{M_{\text{CV}}=M_0}^{M_{\text{CV}}=M_{\text{CV}}} dM_{\text{CV}} = \int_{t=0}^{t=t} \rho\left(V - at\right)A dt$$

$$M_{\text{CV}} - M_0 = \rho\left(Vt - \tfrac{1}{2}at^2\right)A$$

$$M_{\text{CV}} = M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A \quad (6)$$

Now apply the linear momentum equation in the x direction to the same control volume. Note that the frame of reference xy is <u>not</u> inertial since the cart is accelerating.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV \quad (7)$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$ (most of the mass inside the CV has zero velocity in the given frame of reference)

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho\left(V - U\right)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -F$$

$$\int_{\text{CV}} a_{x/X} \rho dV = aM_{\text{CV}}$$

Substitute and re-arrange.

$$-\rho\left(V - U\right)^2 A = -F - aM_{\text{CV}}$$

$$F = \rho\left(V - U\right)^2 A - aM_{\text{CV}} \quad (8)$$

Now substitute Eqns. (3) and (6) into Eqn. (8).

$$\boxed{F = \rho\left(V - at\right)^2 A - a\left[M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A\right]} \quad (9)$$

Now let's solve the problem using a frame of reference fixed to the ground (*XYZ* - inertial).

$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV = \frac{d}{dt}\left(M_{\text{CV}} U\right) = M_{\text{CV}}\frac{dU}{dt} + U\frac{dM_{\text{CV}}}{dt}$$

$$\int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = (V)\left[-\rho(V-U)A\right] = -\rho V (V-U) A$$

$$F_{B,X} = 0$$

$$F_{S,X} = -F$$

Substitute and utilize Eqn. (5) to simplify.

$$M_{\text{CV}}\frac{dU}{dt} + U\frac{dM_{\text{CV}}}{dt} - \rho V(V-U)A = -F$$

$$M_{\text{CV}}\frac{dU}{dt} + U\rho(V-U)A - \rho V(V-U)A = -F$$

$$F = -aM_{\text{CV}} - U\rho(V-U)A + \rho V(V-U)A$$

$$F = \rho(V-U)^2 A - aM_{\text{CV}} \tag{10}$$

Eqn. (10) is identical to Eqn. (8) as expected!