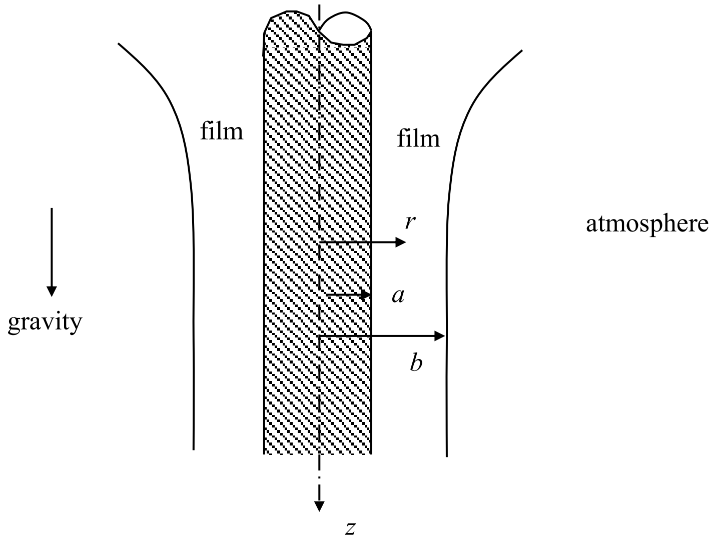
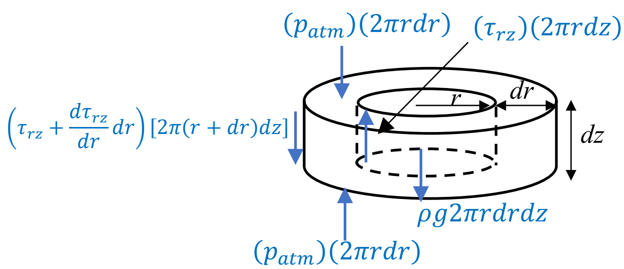
Consider a film of Newtonian liquid draining down the outside of a vertical rod of radius, a, as shown in the figure. Some distance down the rod, a fully developed region is reached where fluid shear balances gravity and the film thickness remains constant. Assuming incompressible, laminar flow and negligible shear interaction with the atmosphere, find an expression for $u_z(r)$ by balancing forces on a differential fluid element.



SOLUTION:



Since the free surface everywhere has pressure p_{atm} and there is no gravity component in the radial direction, the pressure everywhere in the film will be p_{atm}.

Apply a force balance in the z direction to the fluid element shown in the figure,

$$\sum F_z = 0 = \left(\tau_{rz} + \frac{d\tau_{rz}}{dr}dr\right)[2\pi(r+dr)dz] - (\tau_{rz})(2\pi r dz) + (p_{atm})(2\pi r dr) - (p_{atm})(2\pi r dr) + \rho g 2\pi r dr dz, \tag{1}$$

$$\tau_{rz}2\pi r dz + \tau_{rz}2\pi dr dz + \frac{d\tau_{rz}}{dr}dr 2\pi r dz + \frac{d\tau_{rz}}{dr}dr 2\pi dr dz - (\tau_{rz})(2\pi r dz) + \rho g 2\pi r dr dz = 0, \tag{2}$$

$$\left(\tau_{rz} + r\frac{d\tau_{rz}}{dr}\right)2\pi dr dz + \rho g 2\pi r dr dz = 0, \tag{3}$$

$$\left(\tau_{rz} + r\frac{d\tau_{rz}}{dr}\right) + \rho g r = 0, \tag{4}$$

Since the fluid is Newtonian,

$$\tau_{rz} = \mu\frac{du_z}{dr}. \tag{5}$$

Substituting into Eq. (4) and simplifying,

$$\left(\mu\frac{du_z}{dr} + r\mu\frac{d^2u_z}{dr^2}\right) + \rho g r = 0, \tag{6}$$

$$\frac{du_z}{dr} + r\frac{d^2u_z}{dr^2} = -\frac{\rho g}{\mu}r, \tag{7}$$

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = -\frac{\rho g}{\mu}r, \tag{8}$$

$$r\frac{du_z}{dr} = -\frac{\rho g}{2\mu}r^2 + c_1, \tag{9}$$

$$\frac{du_z}{dr} = -\frac{\rho g}{2\mu}r + \frac{c_1}{r}, \tag{10}$$

$$u_z = -\frac{\rho g}{4\mu}r^2 + c_1\ln r + c_2 \tag{11}$$

Now apply boundary conditions to determine the unknown constants,

no-shear at $r = b$: $\tau_{rz}|_{r=b} = \mu \frac{du_z}{dr}\Big|_{r=b} = 0,$ (12)

$\mu \frac{du_z}{dr}\Big|_{r=b} = 0 = -\frac{\rho g}{2} b + \mu \frac{c_1}{b}$ => $c_1 = \frac{\rho g}{2\mu} b^2$ (using Eq. 10). (13)

no-slip at $r = a$: $u_z(r = a) = 0,$ (14)

$0 = -\frac{\rho g}{4\mu} a^2 + \frac{\rho g}{2\mu} b^2 \ln a + c_2$ => $c_2 = \frac{\rho g}{4\mu} a^2 - \frac{\rho g}{2\mu} b^2 \ln a.$ (15)

Thus,

$\boxed{u_z = \frac{\rho g}{4\mu}(a^2 - r^2) + \frac{\rho g}{2\mu} b^2 \ln\left(\frac{r}{a}\right)}.$ (16)

and

$\tau_{rz} = \mu \frac{du_z}{dr} = -\frac{\rho g}{2} r + \frac{\rho g b^2}{2r}.$ (17)