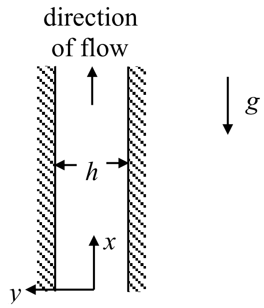
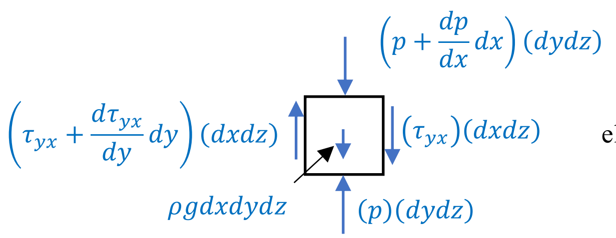
A Newtonian, incompressible fluid flows between the two infinite, vertical, parallel plates shown in the figure. Assume that the flow is steady and fully developed in the x direction and that there is a constant pressure gradient in the x direction, dp/dx.

Determine, by performing a force balance on a differential fluid element, an expression for the velocity component in the x direction.



SOLUTION:



element has dimensions dx, dy, and dz

Apply a force balance in the vertical direction (x-direction) on the differential fluid element shown,

$$\sum F_x = 0 = (p)(dydz) - \left(p + \frac{dp}{dx}dx\right)(dydz) - (\tau_{yx})(dxdz) + \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}dy\right)(dxdz) - \rho g dxdydz, \quad (1)$$

$$-\frac{dp}{dx} + \frac{d\tau_{yx}}{dy} = \rho g. \quad (2)$$

Since the fluid is Newtonian,

$$\tau_{yx} = \mu\frac{du_x}{dy}. \quad (3)$$

Substituting Eq. (3) into Eq. (2) and simplifying,

$$-\frac{dp}{dx} + \mu\frac{d^2u_x}{dy^2} = \rho g, \quad (4)$$

$$\frac{d^2u_x}{dy^2} = \frac{1}{\mu}\left(\rho g + \frac{dp}{dx}\right). \quad (5)$$

Note that since the flow is fully-developed and steady, the pressure gradient dp/dx must not be a function of either y or t. Thus, we can solve the differential equation in Eq. (5) to give,

$$\frac{du_x}{dy} = \frac{1}{\mu}\left(\rho g + \frac{dp}{dx}\right)y + c_1, \quad (6)$$

$$u_x = \frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)y^2 + c_1 y + c_2. \quad (7)$$

The constants c_1 and c_2 can be found from the boundary conditions,

no-slip at the walls: $u_x(y = \pm h/2) = 0 \Rightarrow$ (8)

$$\frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)\frac{h^2}{4} + c_1\frac{h}{2} + c_2 = 0, \quad (9)$$

$$\frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)\frac{h^2}{4} - c_1\frac{h}{2} + c_2 = 0. \quad (10)$$

Add Eqs. (9) and (10),

$$\frac{1}{\mu}\left(\rho g + \frac{dp}{dx}\right)\frac{h^2}{4} + 2c_2 = 0 \Rightarrow c_2 = -\frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)\frac{h^2}{4}. \quad (11)$$

Subtract Eq. (10) from Eq. (9),

$$c_1 h = 0 \Rightarrow c_1 = 0. \quad (12)$$

Thus,

$$u_x = \frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)y^2 - \frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)\frac{h^2}{4}, \quad (13)$$

$$\boxed{u_x = \frac{1}{2\mu}\left(\rho g + \frac{dp}{dx}\right)\left(y^2 - \frac{h^2}{4}\right)}. \quad (14)$$