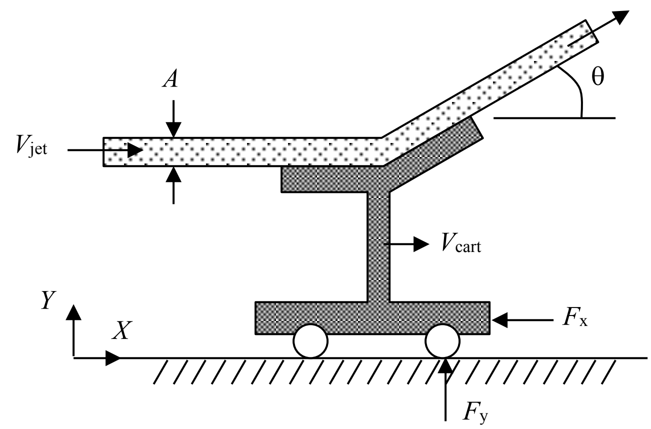
A jet of water is deflected by a vane mounted on a cart. The water jet has an area, A, everywhere and is turned an angle θ with respect to the horizontal. The pressure everywhere within the jet is atmospheric. The incoming jet velocity with respect to the ground (axes XY) is V_{jet}. The cart has mass M. Determine:

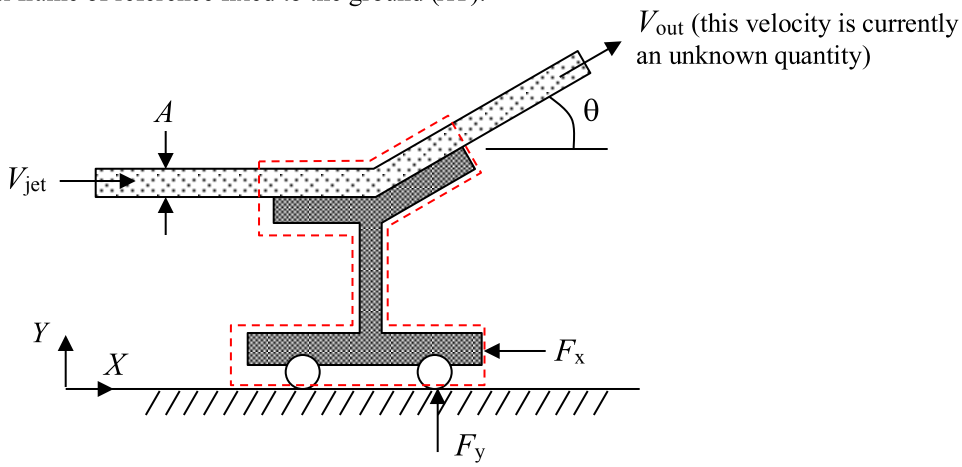
a. the force components, F_x and F_y, required to hold the cart stationary,
b. the horizontal force component, F_x, if the cart moves to the right at the constant velocity, V_{cart} ($V_{cart} < V_{jet}$)



SOLUTION:

Part (a):

Apply conservation of mass and conservation of linear momentum to a control volume surrounding the cart. Use an inertial frame of reference fixed to the ground (XY).



First apply conservation of mass to the control volume to determine V_{out}.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(the mass within the control volume doesn't change)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \underbrace{\left(\rho \overbrace{V_{\text{jet}}\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right)}_{\text{left side}} + \underbrace{\left[\rho \overbrace{V_{\text{out}}\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{A\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho V_{\text{jet}} A + \rho V_{\text{out}} A \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}$$

$$= -\rho V_{\text{jet}} A + \rho V_{\text{out}} A$$

(Note that the jet area remains constant.)

Substitute and re-arrange.

$$-\rho V_{\text{jet}} A + \rho V_{\text{out}} A = 0$$

$$V_{\text{out}} = V_{\text{jet}} \tag{2}$$

Now apply conservation of linear momentum in the X-direction:

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = F_{B,X} + F_{S,X} \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = 0 \quad \text{(the momentum within the control volume doesn't change with time)}$$

$$\int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\left(V_{\text{jet}}\right)}_{=u_X}\underbrace{\left(\rho\underbrace{V_{\text{jet}}\hat{\mathbf{i}}}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{-A\hat{\mathbf{i}}}_{=\mathbf{A}}\right)}_{\text{left side}}+\underbrace{\left(V_{\text{jet}}\cos\theta\right)}_{=u_X}\underbrace{\left[\rho\underbrace{V_{\text{jet}}\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=-\rho V_{\text{jet}}^2A+\rho V_{\text{jet}}^2A\cos\theta\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho V_{\text{jet}}^2A\left(\cos\theta-1\right)$$

$F_{B,X}=0$ (no body forces in the x-direction)

$F_{S,X}=-F_x$ (all of the pressure forces cancel out)

Substitute and re-arrange.

$$\rho V_{\text{jet}}^2A\left(\cos\theta-1\right)=-F_x$$

$$\boxed{F_x=\rho V_{\text{jet}}^2A\left(1-\cos\theta\right)} \tag{4}$$

Now look at the Y-direction:

$$\frac{d}{dt}\int_{CV} u_Y\rho dV+\int_{CS} u_Y\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=F_{B,Y}+F_{S,Y} \tag{5}$$

where

$\frac{d}{dt}\int_{CV} u_Y\rho dV=0$ (the momentum within the control volume doesn't change with time)

$$\int_{CS} u_Y\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\left(V_{\text{jet}}\sin\theta\right)}_{=u_Y}\underbrace{\left[\rho\underbrace{V_{\text{jet}}\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=\rho V_{\text{jet}}^2A\sin\theta\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho V_{\text{jet}}^2A\sin\theta$$

$F_{B,Y}=-Mg$ (assume that the fluid weight in the CV is negligible compared to the cart weight)

$F_{S,Y}=F_y$ (all of the pressure forces cancel out)

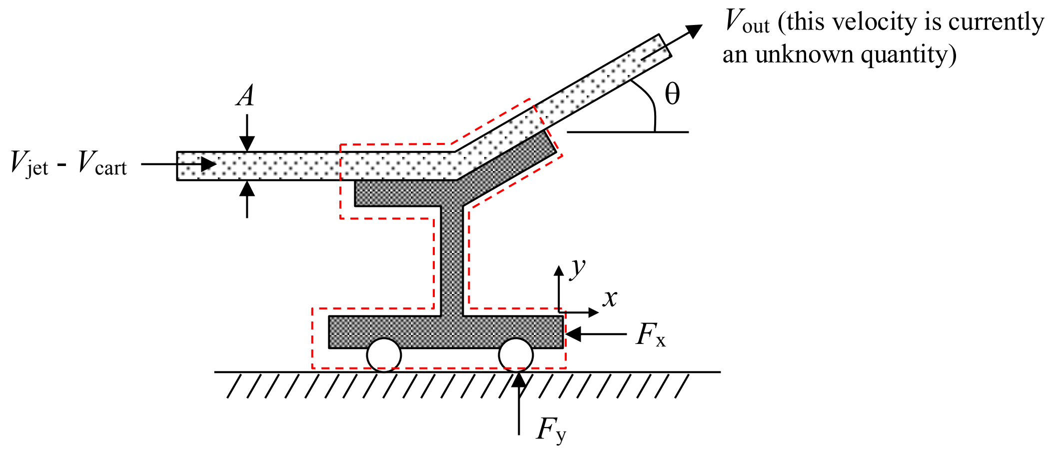
Substitute and re-arrange.

$$\rho V_{\text{jet}}^2A\sin\theta=-Mg+F_y$$

$$\boxed{F_y=\rho V_{\text{jet}}^2A\sin\theta+Mg} \tag{6}$$

Part (b):

Apply conservation of linear momentum to a control volume surrounding the cart. Use a frame of reference fixed to the cart (*xy*). Note that this is an inertial frame of reference since the cart moves in a straight line at a constant speed. In addition, in this frame of reference, the cart appears stationary and the jet velocity at the left is equal to V_{jet}-V_{cart}.



First apply conservation of mass to the control volume to determine V_{out}

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{7}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(the mass within the control volume doesn't change)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \underbrace{\Big[\rho \overbrace{\left(V_{\text{jet}} - V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}} \Big]}_{\text{left side}} + \underbrace{\Big[\rho V_{\text{out}} \overbrace{\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{A\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}} \Big]}_{\text{right side}}$$

$$= -\rho\left(V_{\text{jet}} - V_{\text{cart}}\right)A + \rho V_{\text{out}} A \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}$$

$$= -\rho\left(V_{\text{jet}} - V_{\text{cart}}\right)A + \rho V_{\text{out}} A$$

(Note that the jet area remains constant.)

Substitute and re-arrange.

$$-\rho\left(V_{\text{jet}} - V_{\text{cart}}\right)A + \rho V_{\text{out}} A = 0$$

$$V_{\text{out}} = V_{\text{jet}} - V_{\text{cart}} \tag{8}$$

Now apply conservation of linear momentum in the *x*-direction:

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = F_{B,x} + F_{S,x} \tag{9}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(the momentum within the control volume doesn't change with time)}$$

$$\int_{CS} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)}_{=u_x}\underbrace{\left[\rho\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{-A\hat{\mathbf{i}}}_{=\mathbf{A}}\right]}_{\text{left side}}+\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta}_{=u_X}\underbrace{\left[\rho\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=-\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A+\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\cos\theta\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)$$

$F_{B,x}=0$ (no body forces in the x-direction)

$F_{S,x}=-F_x$ (all of the pressure forces cancel out)

Substitute and re-arrange.

$$\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)=-F_x$$

$$\boxed{F_x=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(1-\cos\theta\right)} \tag{10}$$

Now solve the problem using an inertial frame of reference fixed to the ground (frame XY). From Eqn. (8) we know that using a frame of reference fixed to the cart, the jet velocity on the right hand side is:

$$\mathbf{V}_{\substack{\text{out,}\\ \text{relative to cart}}}=\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right) \tag{11}$$

Hence, relative to the ground the jet velocity on the right hand side is:

$$\mathbf{V}_{\substack{\text{out,}\\ \text{relative to}\\ \text{ground}}}=\mathbf{V}_{\substack{\text{out,}\\ \text{relative to}\\ \text{cart}}}+\mathbf{V}_{\text{cart}}=\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)+V_{\text{cart}}\hat{\mathbf{i}} \tag{12}$$

Now consider conservation of linear momentum in the X direction.

$$\frac{d}{dt}\int_{CV}u_X\rho dV+\int_{CS}u_X\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=F_{B,X}+F_{S,X} \tag{13}$$

where

$\frac{d}{dt}\int_{CV}u_X\rho dV=0$ (the momentum within the control volume doesn't change with time)

$$\int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{V_{\text{jet}}}_{=u_X}\underbrace{\left[\rho\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{-A\hat{\mathbf{i}}}_{=\mathbf{A}}\right]}_{\text{left side}}+\underbrace{\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta+V_{\text{cart}}\right]}_{=u_X}\underbrace{\left[\rho\underbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}}\cdot\underbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}_{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=-\rho V_{\text{jet}}\left(V_{\text{jet}}-V_{\text{cart}}\right)A+\rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta+V_{\text{cart}}\left(V_{\text{jet}}-V_{\text{cart}}\right)\right]A\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho\left[-V_{\text{jet}}^2+V_{\text{jet}}V_{\text{cart}}+\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta+V_{\text{cart}}V_{\text{jet}}-V_{\text{cart}}^2\right]A$$

$$=\rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta-\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\right]A$$

$$=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\left(\cos\theta-1\right)A$$

$F_{B,X}=0$ (no body forces in the x-direction)

$F_{S,X}=-F_x$ (all of the pressure forces cancel out)

Substitute and re-arrange.

$$\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)=-F_x$$

$$\boxed{F_x=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(1-\cos\theta\right)}\quad\text{(Same answer as before!)} \tag{14}$$

Note that using a frame of reference that is fixed to the control volume is easier than using one fixed to the ground. This is often the case.