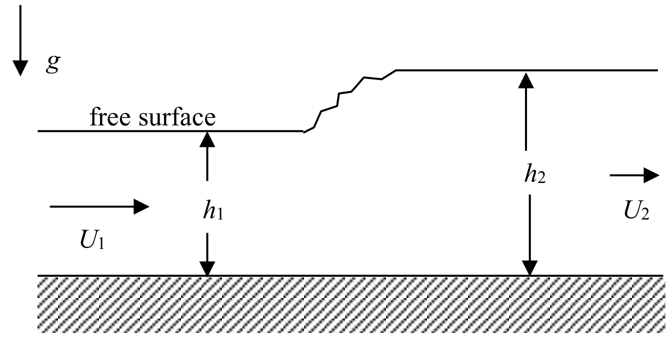
A hydraulic jump is a sudden increase in the depth of a liquid stream (which in this case we assume is flowing over a horizontal stream bed with atmospheric pressure air everywhere above the liquid):



The depth increases suddenly from h_1 to h_2 downstream of the jump. The jump itself is often turbulent and involves viscous losses so that the total pressure downstream is less than that of the upstream flow.

a. Find the ratio of the depths, h_2/h_1, in terms of the upstream velocity, U_1, the depth, h_1, and g, the acceleration due to gravity. Assume the flows upstream and downstream have uniform velocity parallel to the stream bed and that the shear stress between the liquid and the stream bed is zero. The liquid is incompressible.

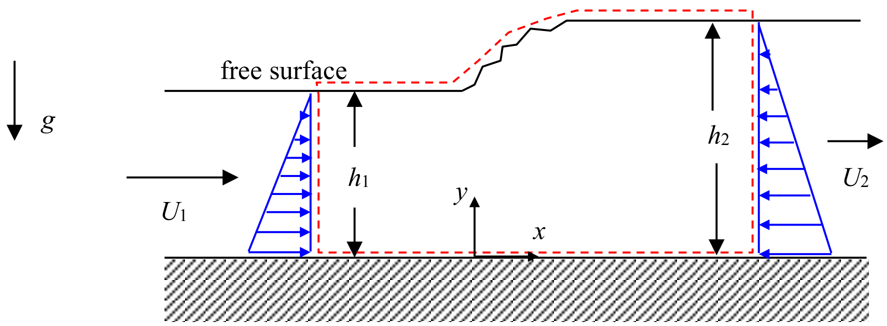
b. What inequality on the value of $U_1^2/(gh_1)$ must hold for a hydraulic jump like this to occur?

BRIEF SOLUTION:

1. Apply conservation of mass to relate the upstream and downstream depths, h_1 and h_2, to the upstream and downstream velocities, U_1 and U_2. Use a control volume that perpendicularly crosses the upstream and downstream flows where the velocities are uniform, follows the free surface, and is adjacent to the floor.
2. Apply the linear momentum equation to the same control volume as in Step 2. Be sure to include the pressure forces acting on the upstream and downstream faces. Note that the pressure increases linearly with depth in the fluid.

DETAILED SOLUTION:

First apply conservation of mass to the fixed control volume shown below.



Note: Since the streamlines are parallel at the inlet and outlet of the CV, the pressure gradient normal to the streamlines will be hydrostatic.

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho U_1 h_1 b + \rho U_2 h_2 b$$

Substitute and simplify.

$$h_2 U_2 = h_1 U_1 \tag{1}$$

Now apply conservation of linear momentum in the x-direction to the same control volume.

$$\frac{d}{dt}\int_{CV} u\rho dV + \int_{CS} u\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = -\rho U_1^2 h_1 b + \rho U_2^2 h_2 b$$

$$F_{B,x} = 0$$

$$F_{S,x} = \int_{y=0}^{y=h_1}\rho g\left(h_1 - y\right)dyb - \int_{y=0}^{y=h_2}\rho g\left(h_2 - y\right)dyb = \tfrac{1}{2}\rho g h_1^2 b - \tfrac{1}{2}\rho g h_2^2 b$$

(hydrostatic pressure forces on left and right sides)

Substitute and simplify making use of Eqn. (1). Solve for the ratio h_2/h_1.

$$-\rho U_1^2 h_1 b + \rho U_2^2 h_2 b = \tfrac{1}{2}\rho g h_1^2 b - \tfrac{1}{2}\rho g h_2^2 b$$

$$-U_1^2 h_1 + U_2^2 h_2 = \tfrac{1}{2} g\left(h_1^2 - h_2^2\right)$$

$$U_1^2 h_1\left({}^{U_2}\!/\!{}_{U_1} - 1\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right) \quad \text{(using Eqn. (1))}$$

$$U_1^2 h_1\left({}^{h_1}\!/\!{}_{h_2} - 1\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right) \quad \text{(using Eqn. (1))}$$

$$U_1^2 h_1\left(\frac{h_1 - h_2}{h_2}\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right)$$

$$U_1^2 \frac{h_1}{h_2} = \tfrac{1}{2} g\left(h_1 + h_2\right) \;\Rightarrow\; \frac{U_1^2}{{}^{h_2}\!/\!{}_{h_1}} = \tfrac{1}{2} g h_1\left(1 + {}^{h_2}\!/\!{}_{h_1}\right)$$

$$\frac{2U_1^2}{g h_1} = {}^{h_2}\!/\!{}_{h_1}\left(1 + {}^{h_2}\!/\!{}_{h_1}\right) \;\Rightarrow\; \left({}^{h_2}\!/\!{}_{h_1}\right)^2 + {}^{h_2}\!/\!{}_{h_1} - \frac{2U_1^2}{g h_1} = 0$$

$${}^{h_2}\!/\!{}_{h_1} = \frac{-1 \pm \sqrt{1 + \dfrac{8U_1^2}{g h_1}}}{2}$$

We can neglect the negative sign in front of the second term since it is unrealistic.

$$\boxed{\therefore\; {}^{h_2}\!/\!{}_{h_1} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1^2}{g h_1}}} \tag{2}$$

For the hydraulic jump to occur, we need $h_2/h_1 > 1$.

$$1 < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1^2}{g h_1}} \;\Rightarrow\; \boxed{\frac{U_1^2}{g h_1} > 1} \tag{3}$$

The dimensionless parameter in Eqn. (3) is the square of the flow's <u>Froude number, Fr</u>.

$$\mathrm{Fr} \equiv \frac{U_1}{\sqrt{g h_1}} \tag{4}$$

where Fr < 1 is referred to as <u>subcritical flow</u>, Fr = 1 is <u>critical flow</u>, and Fr > 1 is <u>supercritical flow</u>. For the hydraulic jump to occur, we must have supercritical flow, *i.e.* Fr > 1.