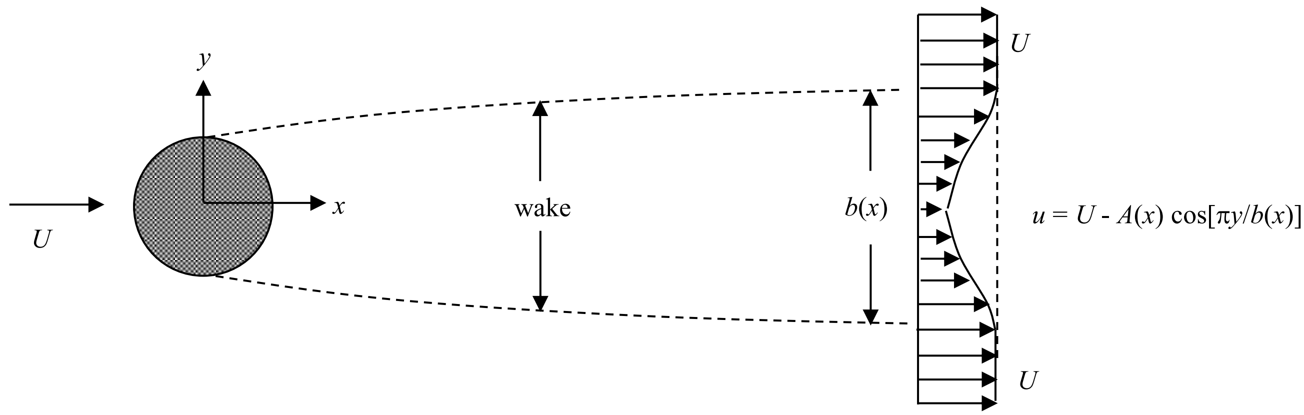
Wake surveys are made in the two-dimensional wake behind a cylindrical body which is externally supported in a uniform stream of incompressible fluid approaching the cylinder with velocity, *U*.



The surveys are made at *x* locations sufficiently far downstream of the body so that the pressure across the wake is the same as the ambient pressure in the fluid far from the body. The surveys indicate that, to a first approximation, the velocity in the wake varies with lateral position, *y*, according to:

$$\frac{u}{U} = 1 - \frac{A(x)}{U}\cos\left[\pi \frac{y}{b(x)}\right] \quad \text{where} \quad -\frac{1}{2} < \frac{y}{b(x)} < +\frac{1}{2}$$

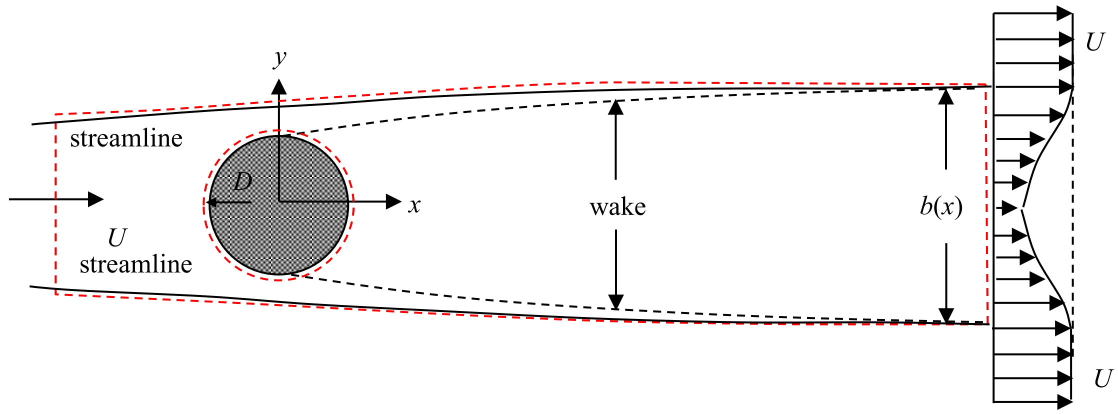
The quantities $A(x)$ and $b(x)$ are the centerline velocity defect and wake width, respectively, both of which vary with position, *x*. If the drag on the body per unit distance normal to the plane of the sketch is denoted by *D* and the density of the fluid by ρ, find the relation for $b(x)$ in terms of $A(x)$, *U*, ρ, and *D*.

BRIEF SOLUTION:

1. First apply the linear momentum equation to determine a relation between the various quantities. Use a control volume that surrounds the cylinder, crosses the flow perpendicularly far upstream of the cylinder where the velocity is uniform (call this cross stream distance, h), crosses the flow perpendicularly downstream of the cylinder where the wake width is $b(x)$, and follows streamlines between the upstream to downstream locations along the sides of the control volume. Note that there is no flow across a streamline. Be sure to include the force the cylinder exerts on the control volume.
2. Apply conservation of mass to the same control volume described above to relate the upstream cross flow width, h, to the downstream cross flow width, $b(x)$.

DETAILED SOLUTION:

Apply linear momentum equation in the *x*-direction to the control volume shown below. Use the fixed frame of reference indicated in the figure. Note that there is no flow across a streamline.



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = U\left(-\rho U h\right) + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} u \rho u dy$$

$$= -\rho U^2 h + \rho U^2 \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \left[1 - \frac{A}{U}\cos\left(\pi \frac{y}{b}\right)\right]^2 dy$$

$$= -\rho U^2 h + \rho U^2 \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \left[1 - \frac{2A}{U}\cos\left(\pi \frac{y}{b}\right) + \frac{A^2}{U^2}\cos^2\left(\pi \frac{y}{b}\right)\right] dy$$

$$= -\rho U^2 h + \rho U^2 \left[b - \underbrace{\frac{2Ab}{\pi U}\sin\left(\pi \frac{y}{b}\right)\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\frac{4Ab}{\pi U}} + \frac{bA^2}{\pi U^2}\left\{ \underbrace{\frac{\pi}{2}\frac{y}{b}\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\pi/2} + \underbrace{\frac{1}{4}\sin\left(2\pi \frac{y}{b}\right)\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=0} \right\} \right]$$

$$= -\rho U^2 h + \rho U^2 \left[b - \frac{4Ab}{\pi U} + \frac{bA^2}{2U^2} \right]$$

$$= \rho U^2 \left[-h + b - \frac{4Ab}{\pi U} + \frac{bA^2}{2U^2} \right]$$

$F_{B,x} = 0$ (no body forces in the *x* direction)

$F_{S,x} = -D$ (no pressure forces in the *x* direction)

Substitute and simplify.

$$\rho U^2\left[-h+b-\frac{4Ab}{\pi U}+\frac{bA^2}{2U^2}\right]=-D \tag{2}$$

Now apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}}\rho dV+\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}}\rho dV=0 \quad \text{(steady flow)}$$

$$\begin{aligned}
\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} &= -\rho Uh+\int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b}\rho u dy \\
&= -\rho Uh+\rho U\int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b}\left[1-\frac{A}{U}\cos\left(\pi\frac{y}{b}\right)\right]dy \\
&= -\rho Uh+\rho U\left[b-\underbrace{\frac{bA}{\pi U}\sin\left(\pi\frac{y}{b}\right)\Big|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\frac{2bA}{\pi U}}\right] \\
&= -\rho Uh+\rho Ub\left[1-\frac{2A}{\pi U}\right]
\end{aligned}$$

Substitute and simplify.

$$0=-\rho Uh+\rho Ub\left[1-\frac{2A}{\pi U}\right]$$

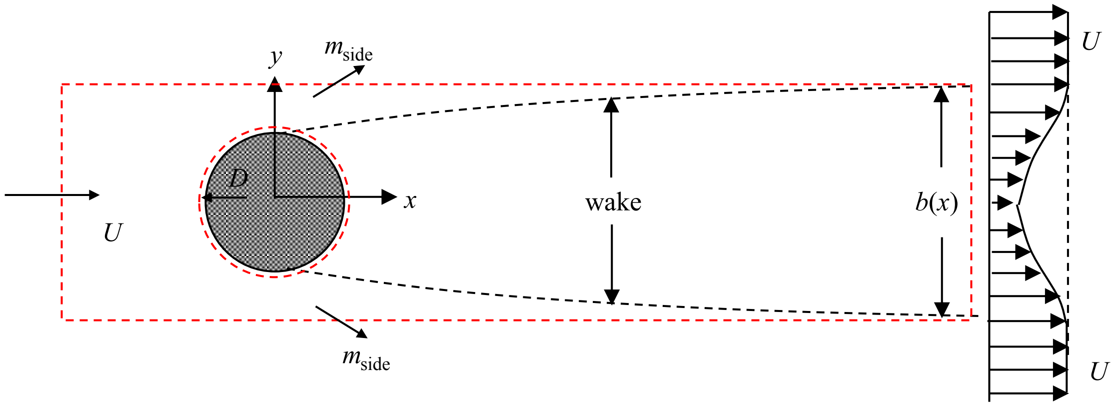
$$h=b\left[1-\frac{2A}{\pi U}\right] \tag{4}$$

Substitute Eqn. (4) into Eqn. (2) and solve for $b(x)$.

$$\rho U^2\left[-b\left(1-\frac{2A}{\pi U}\right)+b-\frac{4Ab}{\pi U}+\frac{bA^2}{2U^2}\right]=-D$$

$$\boxed{\therefore b(x)=\frac{D}{\rho U^2 A(x)\left[\frac{2}{\pi U}-\frac{A(x)}{2U^2}\right]}} \tag{5}$$

The rectangular control volume shown below could also have used. Note that there will be some mass flow rate through the sides as indicated in the figure below (since the upstream mass flux is larger than the downstream mass flux). The horizontal velocity through the sides will be *U* everywhere since the boundaries are outside the wake.



The linear momentum equation in the *x*-direction is:

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{6}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = U\left(-\rho U b\right) + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} u \rho u dy + 2m_{\text{side}} U$$

$$= \rho U^2 \left[-\frac{4Ab}{\pi U} + \frac{bA^2}{2U^2}\right] + 2m_{\text{side}} U$$

$F_{B,x} = 0$ (no body forces in the *x* direction)

$F_{S,x} = -D$ (no pressure forces in the *x* direction)

Substitute and simplify.

$$\rho U^2 \left[-\frac{4Ab}{\pi U} + \frac{bA^2}{2U^2}\right] + 2m_{\text{side}} U = -D \tag{7}$$

Now apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{8}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho U b + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \rho u dy + 2m_{\text{side}}$$

$$= -\rho U b + \rho U b\left[1 - \frac{2A}{\pi U}\right] + 2m_{\text{side}}$$

$$= -\rho b \frac{2A}{\pi} + 2m_{\text{side}}$$

Substitute and simplify.

$$0 = -\rho b \frac{2A}{\pi} + 2m_{\text{side}}$$

$$m_{\text{side}} = \frac{\rho b A}{\pi} \tag{9}$$

Substitute Eqn. (9) into Eqn. (7)and solve for $b(x)$.

$$\therefore b(x) = \frac{D}{\rho U^2 A(x)\left[\frac{2}{\pi U} - \frac{A(x)}{2U^2}\right]} \quad \text{(This is the same result as before!)} \tag{10}$$