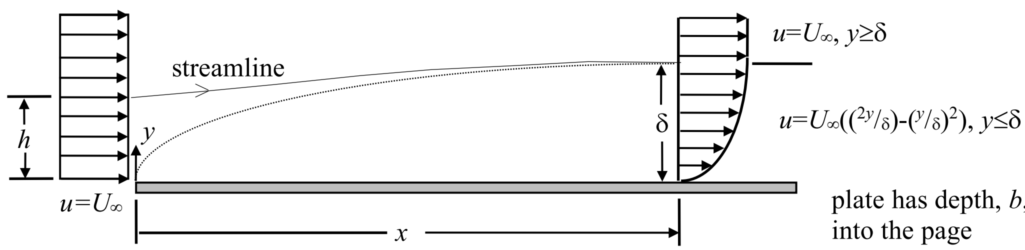
An incompressible, viscous fluid with density, ρ, flows past a solid flat plate which has a depth, *b*, into the page. The flow initially has a uniform velocity U_∞, before contacting the plate. After contact with the plate at a distance *x* downstream from the leading edge, the flow velocity profile is altered due to the no-slip condition. The velocity profile at location *x* is estimated to have a parabolic shape, $u=U_\infty((2y/\delta)-(y/\delta)^2)$, for $y \le \delta$ and $u=U_\infty$ for $y\ge\delta$ where δ is termed the "boundary layer thickness."



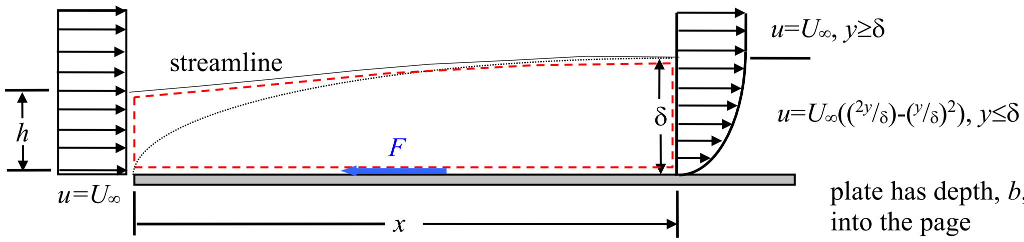
1. Determine the upstream height from the plate, *h*, of a streamline which has a height, δ, at the downstream distance *x*. Express your answer in terms of δ.
2. Determine the force the plate exerts on the fluid over the distance *x*. Express your answer in terms of ρ, U_∞, *b*, and δ. You may assume that the pressure everywhere is p_∞. The force the drag exerts on the plate is called the "skin friction" drag.

BRIEF SOLUTION:

1. Apply conservation of mass to a control volume that is adjacent to the plate, crosses perpendicularly to the stream at the leading edge of the plate, follows a streamline, and crosses perpendicularly to the stream at the location where the boundary layer has thickness, δ. Note that there is no flow across a streamline.
2. Apply the linear momentum equation to the same control volume used in Step 1. Be sure to include the force the plate exerts on the control volume.

DETAILED SOLUTION:

Apply conservation of mass to the fixed control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho U_\infty hb + \int_{y=0}^{y=\delta}\rho U_\infty\left[2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right]dyb = -\rho U_\infty hb + \rho U_\infty\left(\delta - \frac{1}{3}\delta\right)b$$

$$= -\rho U_\infty hb + \tfrac{2}{3}\rho U_\infty \delta b$$

(Note that there is no flow across the streamline.)

Substitute into conservation of mass and solve for h.

$$\boxed{h = \tfrac{2}{3}\delta} \qquad (1)$$

Now apply the linear momentum equation in the x-direction on the same control volume.

$$\frac{d}{dt}\int_{CV}u\rho dV + \int_{CS}u\left(\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV}u\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho U_\infty^2 hb + \int_{y=0}^{y=\delta} \rho U_\infty^2 \left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right]^2 dyb$$

$$= -\rho U_\infty^2 hb + \rho U_\infty^2 b \int_0^\delta \left[4\frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^3} + \frac{y^4}{\delta^4}\right] dy$$

$$= -\rho U_\infty^2 hb + \rho U_\infty^2 b \left[\tfrac{4}{3}\delta - \delta + \tfrac{1}{5}\delta\right]$$

$$= -\rho U_\infty^2 hb + \tfrac{8}{15}\rho U_\infty^2 b\delta$$

$F_{B,x} = 0$

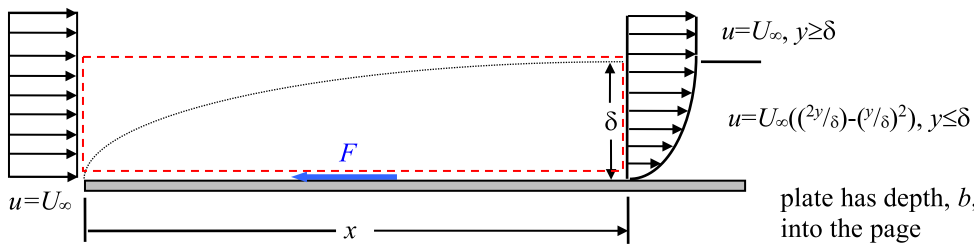
$F_{S,x} = -F$ (the pressure everywhere is p_∞)

Substitute and simplify, making use of Eqn. (1).

$$-\rho U_\infty^2 \left(\tfrac{2}{3}\delta\right) b + \tfrac{8}{15}\rho U_\infty^2 b\delta = -F$$

$$\boxed{F = \tfrac{2}{15}\rho U_\infty^2 b\delta} \quad (2)$$

We could have also determined the force using a different control volume as shown below.



Determine the mass flow rate out of the control volume through the top using conservation of mass.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho U_\infty \delta b + \int_{y=0}^{y=\delta} \rho U_\infty \left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right] dyb + m_{\text{top}} = -\tfrac{1}{3}\rho U_\infty \delta b + m_{\text{top}}$$

Substitute and solve for the mass flow rate.

$$m_{\text{top}} = \tfrac{1}{3}\rho U_\infty \delta b \quad (3)$$

Now apply the linear momentum equation in the *x*-direction to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u\rho dV + \int_{\text{CS}} u\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = -\rho U_\infty^2\delta b + \int_{y=0}^{y=\delta}\rho U_\infty^2\left[2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right]^2 dyb + m_{\text{top}}U_\infty$$

$$= -\tfrac{7}{15}\rho U_\infty^2\delta b + m_{\text{top}}U_\infty$$

(Note that the horizontal component of the velocity at the top is U_∞ since it's outside of the boundary layer.)

$$F_{B,x} = 0$$

$$F_{S,x} = -F \quad \text{(the pressure everywhere is } p_\infty\text{)}$$

Substitute and simplify making use of Eqn. (3).

$$-\tfrac{7}{15}\rho U_\infty^2\delta b + \left(\tfrac{1}{3}\rho U_\infty\delta b\right)U_\infty = -F$$

$$F = \tfrac{2}{15}\rho U_\infty^2\delta b \quad \text{(This is the same answer as before!)}$$