A measured dimensionless laminar boundary layer profile for flow past a flat plate is given in the table below. Use the momentum integral equation to determine the 99% boundary layer thickness. Compare your result with the exact (Blasius) result.

y/δ	u/U
0.00	0.00
0.08	0.133
0.16	0.265
0.24	0.394
0.32	0.517
0.40	0.630
0.48	0.729
0.56	0.811
0.64	0.876
0.72	0.923
0.80	0.956
0.88	0.976
0.96	0.988
1.00	1.000

SOLUTION:

Apply the Kármán Momentum Integral Equation:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient:

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \tag{2}$$

Simplifying Eqn. (1) gives:

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by:

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(y/\delta\right)$$

Integrating the data numerically using the trapezoidal rule gives:

$$\delta_M \approx 0.131\delta \tag{4}$$

Substitute into Eqn. (3).

$$\tau_w = 0.131 \rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as:

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu \frac{U}{\delta} \left.\frac{d\left(u/U\right)}{d\left(y/\delta\right)}\right|_{y/\delta=0} \tag{6}$$

Differentiating the data numerically using a 1st order finite difference scheme:

$$\tau_w \approx 1.66 \mu \frac{U}{\delta} \tag{7}$$

Equating Eqns. (5) and (7) gives:

$$0.131 \rho U^2 \frac{d\delta}{dx} = 1.66 \mu \frac{U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta \, d\delta = 12.67 \frac{\mu}{\rho U} \int_{x=0}^{x=x} dx$$

$$\tfrac{1}{2}\delta^2 = 12.67 \frac{\mu x}{\rho U}$$

$$\boxed{\therefore \frac{\delta}{x} = 5.034 \sqrt{\frac{\mu}{\rho U x}} = \frac{5.034}{\mathrm{Re}_x^{1/2}}} \tag{8}$$

Equation (8) is within 1% of the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

Another approach to this problem is to fit a polynomial curve to the given data rather than numerically differentiating and integrating the data.