A velocity field is given by:

\[
\mathbf{u} = \frac{-V_0y}{(x^2 + y^2)^{\frac{3}{2}}} \mathbf{i} + \frac{V_0x}{(x^2 + y^2)^{\frac{3}{2}}} \mathbf{j}
\]

where \( V_0 \) is a positive constant, \( i.e. \ V_0 > 0 \). Determine:

a. where in the flow the speed is \( V_0 \)
b. the equation and sketch of the streamlines
c. the equations for the streaklines and pathlines

Answer(s):
\[ \mathbf{u} \cdot \mathbf{V} = V_0 \]

The flow speed is everywhere equal to \( V_0 \).

\[ x^2 + y^2 = x_0^2 + y_0^2 = \text{constant} \]

The streamlines are circles! Note that when \( x > 0 \) and \( y > 0 \), \( u_x < 0 \) and \( u_y > 0 \) (note that \( V_0 > 0 \)) so that the flow is moving in a counter-clockwise direction.

Since the flow is steady, the streaklines and pathlines will be identical to the streamlines.
Consider a 2D flow with a velocity field given by:

\[ \mathbf{u} = x(1 + 2t) \hat{i} + y \hat{j} \]

Determine the equations for the streamline, streakline, and pathline passing through the point \((x,y) = (1,1)\) at time \(t = 0\).

**Answer(s):**

Streamline: \( y = x \)

Streakline: \( x = \exp\left(-t_0 - t^2_0\right); \quad y = \exp(-t_0) \)

Pathline: \( x = \exp\left(t + t^2\right); \quad y = \exp(t) \)
One technique for visualizing fluid flow over a surface is to attach short, lightweight pieces of thread or “tufts” to the surface. A photograph of the tufts on the surface of an automobile is shown in the figure below. Do tufts trace out the streamlines, streaklines, pathlines, or some other type of flow line? What if the flow is unsteady? Explain your answers.
A one-dimensional, unsteady velocity field is given by:

\[ \mathbf{u} = U \sin \left( \omega \left( t - \frac{y}{V} \right) \right) \hat{e}_x + V \hat{e}_y \]

where \( U, V, \) and \( \omega \) are positive constants. Find the equations of the streamline, streakline, and pathline that pass through the point \((0, 0)\) at time \( t = 0 \).

**Answer(s):**

Streamline: \( x = \frac{U}{\omega} \left( \omega y + \cos \left( \frac{\omega y}{V} \right) - 1 \right) \)

Streakline: \( x = -Ut_0 \sin \left( \frac{\omega y}{V} \right) ; \quad y = -Vt_0 \)

Pathline: \( x = 0 ; \quad y = Vt \)
Consider the 2D flow field defined by the following velocity:

\[ \mathbf{u} = \left( \frac{1}{1+t} \right) \mathbf{i} + \mathbf{j} \]

For this flow field, find the equation of:

a. the streamline through the point \((1,1)\) at \(t = 0\),

b. the pathline for a particle released at the point \((1,1)\) at \(t = 0\), and

c. the streakline at \(t = 0\) which passes through the point \((1,1)\).

**Answer(s):**

For the streamline passing through the point \((x_0, y_0) = (1, 1)\) at time \(t = 0\):

\[ y = x \]

**Streakline:**

\[ x - 1 = \ln \left( \frac{1}{1+t_0} \right) \quad \Rightarrow \quad x = \ln \left( \frac{1}{1+t_0} \right) + 1 \]

\[ y - 1 = -t_0 \quad \Rightarrow \quad y = 1 - t_0 \]

**Pathline:**

\[ x - 1 = \ln (1+t) \quad \Rightarrow \quad x = \ln (1+t) + 1 \]

\[ y - 1 = t \quad \Rightarrow \quad y = 1 + t \]
Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin \((x = 0, y = 0)\). The velocity field is unsteady and obeys the equations:

\[
\begin{align*}
u &= 1 \text{ m/s} & \quad & v = 1 \text{ m/s} & \quad & 0 \leq t < 2 \text{ s} \\
u &= 0 & \quad & v = 1.5 \text{ m/s} & \quad & 2 \leq t \leq 4 \text{ s}
\end{align*}
\]

Plot the pathlines of bubbles that leave the origin at \(t = 0, 1, 2, 3,\) and \(4 \text{ s}\). Mark the locations of these five bubbles at \(t = 4 \text{ s}\). Use a dashed line to indicate the position of the streakline passing through \((0, 0)\) at \(t = 4 \text{ s}\). What does the streamline passing through \((0, 0)\) look like at \(t = 4 \text{ s}\)?

**Answer(s):**
The streamline passing through \((0, 0)\) at \(t = 4 \text{ s}\) (or any other point for that matter) will be a vertical line since the velocity at \(t = 4 \text{ s}\) is purely vertical.
A tornado can be represented in polar coordinates by the velocity field,

\[ \mathbf{u} = -\frac{a}{r} \hat{r} + \frac{b}{r} \hat{\theta} \]

where \( \hat{r} \) and \( \hat{\theta} \) are unit vectors pointing in the radial (\( r \)) and tangential (\( \theta \)) directions, respectively, and \( a \) and \( b \) are constants. Show that the streamlines for this flow form logarithmic spirals, i.e.

\[ r = c \exp\left( -\frac{a}{b} \theta \right) \]

where \( c \) is a constant.

\[ \therefore r = c \exp\left( -\frac{a}{b} \theta \right) \]
Show that for a steady flow, streamlines, streaklines, and pathlines are identical.

*Answer(s):*

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In addition to the horizontal velocity components of the air in the atmosphere (“wind”), there often are vertical air currents (“thermals”) caused by buoyant effects due to the uneven heating of the air. Assume that the velocity field in a certain region of the atmosphere may be approximated by:
\[ u_y = U \quad \text{for} \quad y > 0 \]
and
\[ u_y = \begin{cases} V \left( 1 - \frac{y}{H} \right) & 0 < y < H \\ 0 & y \geq H \end{cases} \]
Plot the shape of the streamline that passes through the origin for values of \( U/V = 0.5, 1, \) and 2.

**Answer(s):**
\[ \frac{y}{H} = 1 - \exp \left( -\frac{V}{U} \frac{x}{H} \right) \quad \text{for} \quad y < H \]
For \( y \geq H \) the streamlines will be horizontal since \( u_y = 0 \) and \( u_x = U \).