On Delayed Observers for Linear Systems with Unknown Inputs

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Motivation

Consider a linear system $S$ of the form

$$
x_{k+1} = Ax_k + Bu_k
$$

$$
y_k = Cx_k + Du_k
$$

with state vector $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$, and unknown input $u \in \mathbb{R}^m$.

Unknown inputs can be used to model
- Faults
- Parameter uncertainties
- Noise with unknown statistics
- Control inputs generated by controllers in decentralized control

Can we (asymptotically) estimate the state of the system using only the outputs?

Note: known inputs are easy to handle, so we omit them.
Previous Work

- Problem has been investigated extensively over the past few decades
  - Wang, Davison, Dorato, Hautus, Kudva, Viswanadham, Ramakrishna, Dorouach, Zasadzinski, Xu, Hou, Muller, Patton, Yang, Wilde, Valcher, …

- These investigations typically focus on zero-delay observers
  - i.e., use $y_k$ to estimate $x_k$

- Existence conditions for such observers are quite strict

- Conditions can be relaxed by using delayed outputs

- Here, we present a design procedure for reduced-order observers with delays
  - Allows us to treat full-order observers as a special case
Output of system over $\alpha + 1$ time-steps is

\[
\begin{bmatrix}
  y_k \\
  y_{k+1} \\
  \vdots \\
  y_{k+\alpha}
\end{bmatrix}
= \begin{bmatrix}
  C \\
  CA \\
  \vdots \\
  CA^\alpha
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  \Theta_\alpha
\end{bmatrix}
+ \begin{bmatrix}
  D & 0 & \cdots & 0 \\
  CB & D & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  CA^{\alpha-1}B & CA^{\alpha-2}B & \cdots & D
\end{bmatrix}
\begin{bmatrix}
  u_k \\
  u_{k+1} \\
  \vdots \\
  u_{k+\alpha}
\end{bmatrix}
\]
Directly Measurable States

What states can we directly measure from the output over $\alpha + 1$ time-steps?

- Let $\beta_\alpha = \text{rank} \begin{bmatrix} \Theta_\alpha & M_\alpha \end{bmatrix} - \text{rank} \begin{bmatrix} M_\alpha \end{bmatrix}$

- **Theorem:** There are $\beta_\alpha$ linear functionals of the state that are directly available from the output.

- **Proof:**
  - There are $\beta_\alpha$ linearly independent columns in the matrix $\Theta_\alpha$ that cannot be written as a linear combination of columns in $M_\alpha$.
  - Thus there exists a matrix $\mathcal{P}$ with $\beta_\alpha$ rows such that $\mathcal{P}M_\alpha = 0$ and $\mathcal{P}\Theta_\alpha$ has full row-rank, which gives

  $$\mathcal{P}Y_{k:k+\alpha} = \mathcal{P}\Theta_\alpha x_k + \mathcal{P}M_\alpha U_{k:k+\alpha}$$
  $$= \mathcal{P}\Theta_\alpha x_k$$
Observing the Other States

- Choose a matrix $\mathcal{H}$ so that $T \equiv \begin{bmatrix} \mathcal{P}\Theta_\alpha \\ \mathcal{H} \end{bmatrix}$ is square and invertible

- Consider an observer of the form

$$
\begin{align*}
  z_{k+1} &= Ez_k + FY_{k:k+\alpha} , \\
  \psi_k &= z_k + GY_{k:k+\alpha} ,
\end{align*}
$$

where $E$, $F$ and $G$ are chosen so that $\psi_k \to \mathcal{H}x_k$ as $k \to \infty$

- An estimate of the original states can then be obtained as

$$
\hat{x}_k = T^{-1} \begin{bmatrix} \mathcal{P}Y_{k:k+\alpha} \\ \psi_k \end{bmatrix} \to T^{-1} \begin{bmatrix} \mathcal{P}\Theta_\alpha x_k \\ \mathcal{H}x_k \end{bmatrix} = x_k
$$

- Can obtain a full-order observer by choosing $\mathcal{P}$ to be the empty matrix and $\mathcal{H} = I_n$
Observer Design (1)

How do we choose $E$, $F$ and $G$?

- The observer error is given as

$$e_{k+1} \equiv \psi_{k+1} - \mathcal{H} x_{k+1}$$

$$= E z_k + F Y_{k:k+\alpha} + G Y_{k+1:k+\alpha+1} - \mathcal{H} A x_k - \mathcal{H} B u_k$$

- Partition $F$ and $G$ as

$$F = \begin{bmatrix} F_0 & F_1 & \cdots & F_\alpha \end{bmatrix},$$

$$G = \begin{bmatrix} G_0 & G_1 & \cdots & G_\alpha \end{bmatrix}$$

where each $F_i$ and $G_i$ are of dimension $(n - \beta) \times p$

- Define $K \equiv \begin{bmatrix} F_0 - E G_0 & F_1 - E G_1 + G_0 & \cdots & F_\alpha - E G_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix}$
Observer Design (2)

After some algebra, observer error can be written as

\[ e_{k+1} = Ee_k + \left( \begin{bmatrix} 0 & E \end{bmatrix} - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} + K \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \right) T x_k \]

\[ \quad + \left( KM_{\alpha+1} - \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \right) U_{k:k+\alpha+1} \]

where \( \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \equiv \mathcal{H}AT^{-1} \) and \( \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \equiv \Theta_{\alpha+1}T^{-1} \)

To force the error to go to zero, we need:

- Input decoupling: \( KM_{\alpha+1} - \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} = 0 \)

- State decoupling:

\[ 0 = A_{21} - K\Phi_1 \]

\[ E = A_{22} - K\Phi_2 \]

\( E \) must be a stable matrix
Input Decoupling

\[ KM_{\alpha+1} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \]  \hspace{1cm} (1)

**Theorem:** There exists a matrix \( K \) satisfying (1) if and only if

\[
\text{rank } [M_{\alpha+1}] - \text{rank } [M_{\alpha}] = m.
\]

- This is the Massey-Sain condition for system inversion with delay \( \alpha + 1 \) (1969)
  - We must invert the inputs in order to estimate the states
- The larger the delay, the better the chance of satisfying the condition
- Upper bound on inversion delay provided by Willsky (1974) as \( \alpha = n - \text{nullity}[D] \)
The gain $K$ must simultaneously satisfy

$$KM_{\alpha+1} = \begin{bmatrix} HB & 0 \end{bmatrix} \quad \text{(input decoupling)}$$

$$K\Phi_1 = A_{21} \quad \text{(state decoupling)}$$

To satisfy the above, we show that $K$ can be parametrized as

$$K = \begin{bmatrix} L_1 - \hat{K}L_2 & \hat{K} & HB \end{bmatrix} J,$$

for some matrices $L_1$, $L_2$ and $J$

- $\hat{K}$ is a free matrix
Stability (1)

- The second state-decoupling condition was
  \[ E = A_{22} - K \Phi_2 \]

- Using the parametrization of \( K \), we get
  \[ E = A_{22} - \left[ L_1 - \hat{K} L_2 \quad \hat{K} \quad \mathcal{H} B \right] J \Phi_2 \]

- We show that \( J \Phi_2 = \begin{bmatrix} 0 \\ \nu \end{bmatrix} \) for some matrix \( \nu \)

- This leads to
  \[ E = A_{22} - \left[ \hat{K} \quad \mathcal{H} B \right] \nu \]
  \[ \equiv A_{22} - \left[ \hat{K} \quad \mathcal{H} B \right] \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \]
  \[ = (A_{22} - \mathcal{H} B \nu_2) - \hat{K} \nu_1 \]
For $E$ to be stable, $(A_{22} - HB_{2}, \nu_1)$ must be detectable

**Theorem:** The pair $(A_{22} - HB_{2}, \nu_1)$ is detectable if and only if
\[
\text{rank} \left[ \begin{array}{cc} zI - A & -B \\ C & D \end{array} \right] = n + m, \quad \forall z \in \mathbb{C}, \quad |z| \geq 1.
\]

Putting this together with the inversion condition, we get

**Theorem:** The system $S$ has an observer with delay $\alpha$ if and only if
1. $\text{rank} [M_{\alpha+1}] - \text{rank} [M_{\alpha}] = m$,
2. $\text{rank} \left[ \begin{array}{cc} zI - A & -B \\ C & D \end{array} \right] = n + m, \quad \forall z \in \mathbb{C}, \quad |z| \geq 1.$
In fact, the second condition is equivalent to the existence of a stable inverse (Moylan, 1977)

Thus, we get

**Theorem:** The system $S$ has a (delayed) observer if and only if

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \quad \forall z \in \mathbb{C}, \ |z| \geq 1.$$
Final Observer Equations

- Choose $\hat{K}$ to make $E = (A_{22} - \mathcal{H}B\nu_2) - \hat{K}\nu_1$ stable

- Set $K = \begin{bmatrix} L_1 - \hat{K}L_2 & \hat{K} & \mathcal{H}B \end{bmatrix} \mathcal{J}$

- Map this $K$ matrix back to $F$ and $G$ via
  
  $$K \equiv \begin{bmatrix} F_0 - EG_0 & F_1 - EG_1 + G_0 & \cdots & F_\alpha - EG_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix}$$

  - Mapping is not unique

- Final observer given by

  $$z_{k+1} = Ez_k + FY_{k:k+\alpha},$$

  $$\psi_k = z_k + GY_{k:k+\alpha}$$
Consider the system given by the matrices

\[
A = \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{1}{2} & 1 & -2 \\
0 & 0 & 0 & 1 \\
0 & 0 & -\frac{1}{2} & \frac{3}{2}
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & -1 \\
0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & -1 & 0 & -3 \\
0 & 1 & 0 & 2 \\
0 & 1 & -1 & 4
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{bmatrix}
\]

- We find that \( \text{rank}[M_2] - \text{rank}[M_1] = 2 \), so observer must have a minimum delay of \( \alpha = 1 \)
- We have \( \beta_1 = \text{rank} \left[ \Theta_1 \ M_1 \right] - \text{rank}[M_1] = 3 \)
  - Can obtain three linear functionals directly from the output
Example (2)

- Design matrices are chosen as

\[
P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
H = \begin{bmatrix} 1 & -2 & 2 & 1 \end{bmatrix}
\]

- These matrices give us

\[
T = P \Theta_1 H = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & -2 & 1 & -7 \\ 0 & \frac{5}{2} & 0 & 5 \\ 1 & -2 & 2 & 1 \end{bmatrix}
\]
Example (3)

We find

\[ \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} = \mathcal{H} \mathcal{A} \mathcal{T}^{-1} = \begin{bmatrix} \frac{9}{2} & -4 & -\frac{8}{5} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{10} & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} = \Theta_{\alpha+1} \mathcal{T}^{-1} = \begin{bmatrix} 0 & 0 & \frac{2}{10} & 0 \\ 1 & -1 & -\frac{2}{10} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 1 & -1 & -\frac{3}{10} & 0 \end{bmatrix} \]

Note that \( \Phi_2 = 0 \)
Example (4)

The matrix $K$ is parametrized as

$$K = \begin{bmatrix} L_1 - \hat{K} L_2 & \hat{K} & \mathcal{H} B \end{bmatrix} \mathcal{J},$$

where

$$L_1 = \begin{bmatrix} \frac{11}{2} & -\frac{13}{2} & -2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & -1 & 0 & 0 & 0
\end{bmatrix}.$$
Calculate $\nu_1$ and $\nu_2$:

$$
\begin{bmatrix}
0 \\
\nu_1 \\
\nu_2
\end{bmatrix} = \mathcal{J} \Phi_2 = 0
$$

Matrix $E$ is given by $E = (A_{22} - \mathcal{H}B\nu_2) - \hat{K}\nu_1 = \frac{1}{2}$

- $E$ has magnitude less than 1, so observer will be stable
- Choose $\hat{K} = 0$

Gain matrix given by

$$
K = \begin{bmatrix}
L_1 - \hat{K}L_2 & \hat{K} & \mathcal{H}B
\end{bmatrix} \mathcal{J}
$$

$$
= \begin{bmatrix}
\frac{11}{2} & -4 & 6 & -7 & -2 & -2 & 2 & 0 & 0
\end{bmatrix}
$$
Example (6)

- Obtain $F$ and $G$ by choosing $G_0 = 0$

\[
F = \begin{bmatrix} F_0 & F_1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} & -4 & 6 & -6 & -2 & -2 \\ \end{bmatrix}
\]

\[
G = \begin{bmatrix} G_0 & G_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ \end{bmatrix}
\]

- Final observer given by

\[
z_{k+1} = \frac{1}{2} z_k + F Y_{k:k+1}
\]

\[
\psi_k = z_k + G Y_{k:k+1}
\]

- Estimate of original state given by

\[
\hat{x}_k = T^{-1} \begin{bmatrix} P Y_{k:k+1} \\
\psi_k \end{bmatrix}
\]
Example (7)
Summary

- Provided a design procedure for delayed observers for linear systems with unknown inputs
  - Focused on reduced-order observers, allowing full-order observers as special case
- System inversion is necessary in order to construct an observer
  - Characterized the minimum and maximum delays required for state estimation
- Provided a parametrization of the observer gain to perform state and input decoupling
  - Remaining freedom used to ensure stability