On Delayed Observers for Linear Systems with Unknown Inputs

Shreyas Sundaram and Christoforos N. Hadjicostis

Coordinated Science Laboratory

and

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

Motivation

- Consider a linear system $\mathcal{S}$ of the form

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k \quad ,
\end{aligned}
$$

  with state vector $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$, and **unknown** input $u \in \mathbb{R}^m$
- Unknown inputs can be used to model
  - Faults
  - Parameter uncertainties
  - Noise with unknown statistics
  - Control inputs generated by controllers in decentralized control
- Can we (asymptotically) estimate the state of the system using only the outputs?
- Note: known inputs are easy to handle, so we omit them

Previous Work

- Problem has been investigated extensively over the past few decades
  - Wang, Davison, Dorato, Hautus, Kudva, Viswanadham, Ramakrishna, Darouach, Zasadzinski, Xu, Hou, Muller, Patton, Yang, Wilde, Valcher, . . .
- These investigations typically focus on **zero-delay** observers
  - i.e., use y_k to estimate x_k
- Existence conditions for such observers are quite strict
- Conditions can be relaxed by using delayed outputs
  - Jin and Tahk (1997), Saberi, Stoorvogel and Sannuti (2000)
- Here, we present a design procedure for reduced-order observers with delays
  - Allows us to treat full-order observers as a special case

Preliminaries

- Output of system over $\alpha + 1$ time-steps is

$$\underbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+\alpha} \end{bmatrix}}_{Y_{k:k+\alpha}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha} \end{bmatrix}}_{\Theta_\alpha} x_k + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1}B & CA^{\alpha-2}B & \cdots & D \end{bmatrix}}_{M_\alpha} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+\alpha} \end{bmatrix}}_{U_{k:k+\alpha}}$$

Directly Measurable States

What states can we directly measure from the output over $\alpha + 1$ time-steps?

- Let $\beta_\alpha = \text{rank}\begin{bmatrix}\Theta_\alpha & M_\alpha\end{bmatrix} - \text{rank}\begin{bmatrix}M_\alpha\end{bmatrix}$
- **Theorem:** There are β_α linear functionals of the state that are directly available from the output.
- **Proof:**
  - There are β_α linearly independent columns in the matrix Θ_α that cannot be written as a linear combination of columns in M_α
  - Thus there exists a matrix $\mathcal{P}$ with β_α rows such that $\mathcal{P}M_\alpha = \mathbf{0}$ and $\mathcal{P}\Theta_\alpha$ has full row-rank, which gives

$$\begin{aligned}\mathcal{P}Y_{k:k+\alpha} &= \mathcal{P}\Theta_\alpha x_k + \mathcal{P}M_\alpha U_{k:k+\alpha} \\ &= \mathcal{P}\Theta_\alpha x_k\end{aligned}$$

Observing the Other States

- Choose a matrix $\mathcal{H}$ so that $\mathcal{T} \equiv \begin{bmatrix} \mathcal{P}\Theta_\alpha \\ \mathcal{H} \end{bmatrix}$ is square and invertible
- Consider an observer of the form

$$z_{k+1} = Ez_k + FY_{k:k+\alpha} \quad ,$$
$$\psi_k = z_k + GY_{k:k+\alpha} \quad ,$$

  where E, F and G are chosen so that $\psi_k \to \mathcal{H}x_k$ as $k \to \infty$
- An estimate of the original states can then be obtained as

$$\hat{x}_k = \mathcal{T}^{-1} \begin{bmatrix} \mathcal{P}Y_{k:k+\alpha} \\ \psi_k \end{bmatrix} \to \mathcal{T}^{-1} \begin{bmatrix} \mathcal{P}\Theta_\alpha x_k \\ \mathcal{H}x_k \end{bmatrix} = x_k$$

- Can obtain a full-order observer by choosing $\mathcal{P}$ to be the empty matrix and $\mathcal{H} = I_n$

Observer Design (1)

How do we choose E, F and G?

- The observer error is given as

$$\begin{aligned} e_{k+1} &\equiv \psi_{k+1} - \mathcal{H}x_{k+1} \\ &= Ez_k + FY_{k:k+\alpha} + GY_{k+1:k+\alpha+1} - \mathcal{H}Ax_k - \mathcal{H}Bu_k \end{aligned}$$

- Partition F and G as

$$F = \begin{bmatrix} F_0 & F_1 & \cdots & F_\alpha \end{bmatrix} \, ,$$
$$G = \begin{bmatrix} G_0 & G_1 & \cdots & G_\alpha \end{bmatrix}$$

  where each F_i and G_i are of dimension $(n-\beta) \times p$

- Define $K \equiv \begin{bmatrix} F_0 - EG_0 & F_1 - EG_1 + G_0 & \cdots & F_\alpha - EG_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix}$

Observer Design (2)

- After some algebra, observer error can be written as

$$e_{k+1} = Ee_k + \left(\begin{bmatrix}0 & E\end{bmatrix} - \begin{bmatrix}A_{21} & A_{22}\end{bmatrix} + K\begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix}\right)\mathcal{T}x_k$$
$$+ \left(KM_{\alpha+1} - \begin{bmatrix}\mathcal{H}B & 0\end{bmatrix}\right)U_{k:k+\alpha+1}$$

  where $\begin{bmatrix}A_{21} & A_{22}\end{bmatrix} \equiv \mathcal{H}A\mathcal{T}^{-1}$ and $\begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix} \equiv \Theta_{\alpha+1}\mathcal{T}^{-1}$

- To force the error to go to zero, we need:
  - Input decoupling: $KM_{\alpha+1} - \begin{bmatrix}\mathcal{H}B & 0\end{bmatrix} = 0$
  - State decoupling:

$$0 = A_{21} - K\Phi_1$$
$$E = A_{22} - K\Phi_2$$

  - E must be a stable matrix

Input Decoupling

$$KM_{\alpha+1} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \tag{1}$$

Theorem: There exists a matrix K satisfying (1) if and only if

$$\text{rank}\,[M_{\alpha+1}] - \text{rank}\,[M_\alpha] = m \ .$$

- This is the Massey-Sain condition for system inversion with delay $\alpha + 1$ (1969)
  - We must invert the inputs in order to estimate the states
- The larger the delay, the better the chance of satisfying the condition
- Upper bound on inversion delay provided by Willsky (1974) as $\alpha = n - \text{nullity}[D]$

Parametrizing the Gain

- The gain K must simultaneously satisfy

$$K M_{\alpha+1} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \qquad \text{(input decoupling)}$$

$$K \Phi_1 = A_{21} \qquad \text{(state decoupling)}$$

- To satisfy the above, we show that K can be parametrized as

$$K = \begin{bmatrix} L_1 - \widehat{K} L_2 & \widehat{K} & \mathcal{H}B \end{bmatrix} \mathcal{J} \ ,$$

  for some matrices L_1, L_2 and $\mathcal{J}$

  - $\widehat{K}$ is a free matrix

Stability (1)

- The second state-decoupling condition was

$$E = A_{22} - K\Phi_2$$

- Using the parametrization of K, we get

$$E = A_{22} - \begin{bmatrix} L_1 - \widehat{K}L_2 & \widehat{K} & \mathcal{H}B \end{bmatrix} \mathcal{J}\Phi_2$$

- We show that $\mathcal{J}\Phi_2 = \begin{bmatrix} 0 \\ \nu \end{bmatrix}$ for some matrix ν
- This leads to

$$\begin{aligned} E &= A_{22} - \begin{bmatrix} \widehat{K} & \mathcal{H}B \end{bmatrix} \nu \\ &\equiv A_{22} - \begin{bmatrix} \widehat{K} & \mathcal{H}B \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \\ &= (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}\nu_1 \end{aligned}$$

Stability (2)

- For E to be stable, $(A_{22} - \mathcal{H}B\nu_2, \nu_1)$ must be detectable

Theorem: The pair $(A_{22} - \mathcal{H}B\nu_2, \nu_1)$ is detectable if and only if

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1 \;\; .$$

- Putting this together with the inversion condition, we get

Theorem: The system $\mathcal{S}$ has an observer with delay α if and only if

1. rank $[M_{\alpha+1}] -$ rank $[M_\alpha] = m$,
2. rank $\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1 \;\; .$

Stable Inversion

- In fact, the second condition is equivalent to the existence of a stable inverse (Moylan, 1977)
- Thus, we get

Theorem: The system $\mathcal{S}$ has a (delayed) observer if and only if

$$\text{rank}\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1 \;\; .$$

Final Observer Equations

- Choose $\widehat{K}$ to make $E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}\nu_1$ stable
- Set $K = \begin{bmatrix} L_1 - \widehat{K}L_2 & \widehat{K} & \mathcal{H}B \end{bmatrix} \mathcal{J}$
- Map this K matrix back to F and G via

  $K \equiv \begin{bmatrix} F_0 - EG_0 & F_1 - EG_1 + G_0 & \cdots & F_\alpha - EG_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix}$
  - Mapping is not unique
- Final observer given by

$$z_{k+1} = Ez_k + FY_{k:k+\alpha} \ ,$$
$$\psi_k = z_k + GY_{k:k+\alpha}$$

Example (1)

Consider the system given by the matrices

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

- We find that rank$[M_2]$ - rank$[M_1] = 2$, so observer must have a minimum delay of $\alpha = 1$
- We have $\beta_1 =$ rank $\begin{bmatrix} \Theta_1 & M_1 \end{bmatrix}$ $-$ rank$[M_1] = 3$
  - Can obtain three linear functionals directly from the output

Example (2)

- Design matrices are chosen as

$$\mathcal{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathcal{H} = \begin{bmatrix} 1 & -2 & 2 & 1 \end{bmatrix}$$

  - These matrices give us

$$\mathcal{T} = \begin{bmatrix} \mathcal{P}\Theta_1 \\ \mathcal{H} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & -2 & 1 & -7 \\ 0 & \frac{5}{2} & 0 & 5 \\ 1 & -2 & 2 & 1 \end{bmatrix}$$

Example (3)

- We find

$$\left[\begin{array}{c|c} A_{21} & A_{22} \end{array}\right] = \mathcal{H}A\mathcal{T}^{-1} = \left[\begin{array}{ccc|c} \frac{9}{2} & -4 & -\frac{8}{5} & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{c|c} \Phi_1 & \Phi_2 \end{array}\right] = \Theta_{\alpha+1}\mathcal{T}^{-1} = \left[\begin{array}{ccr|c} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{10} & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{10} & 0 \\ 1 & -1 & -\frac{2}{10} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 1 & -1 & -\frac{3}{10} & 0 \end{array}\right]$$

  - Note that $\Phi_2 = 0$

Example (4)

- The matrix K is parametrized as

$$K = \left[L_1 - \widehat{K} L_2 \quad \widehat{K} \quad \mathcal{H}B \right] \mathcal{J} \ ,$$

where

$$L_1 = \begin{bmatrix} \frac{11}{2} & -\frac{13}{2} & -2 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

Example (5)

- Calculate ν_1 and ν_2:

$$\begin{bmatrix} 0 \\ \nu_1 \\ \nu_2 \end{bmatrix} = \mathcal{J}\Phi_2 = 0$$

- Matrix E is given by $E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}\nu_1 = \frac{1}{2}$
  - E has magnitude less than 1, so observer will be stable
  - Choose $\widehat{K} = 0$
- Gain matrix given by

$$\begin{aligned} K &= \begin{bmatrix} L_1 - \widehat{K}L_2 & \widehat{K} & \mathcal{H}B \end{bmatrix} \mathcal{J} \\ &= \begin{bmatrix} \frac{11}{2} & -4 & 6 & -7 & -2 & -2 & 2 & 0 & 0 \end{bmatrix} \end{aligned}$$

Example (6)

- Obtain F and G by choosing $G_0 = 0$

$$F = \begin{bmatrix} F_0 & F_1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} & -4 & 6 & -6 & -2 & -2 \end{bmatrix}$$
$$G = \begin{bmatrix} G_0 & G_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

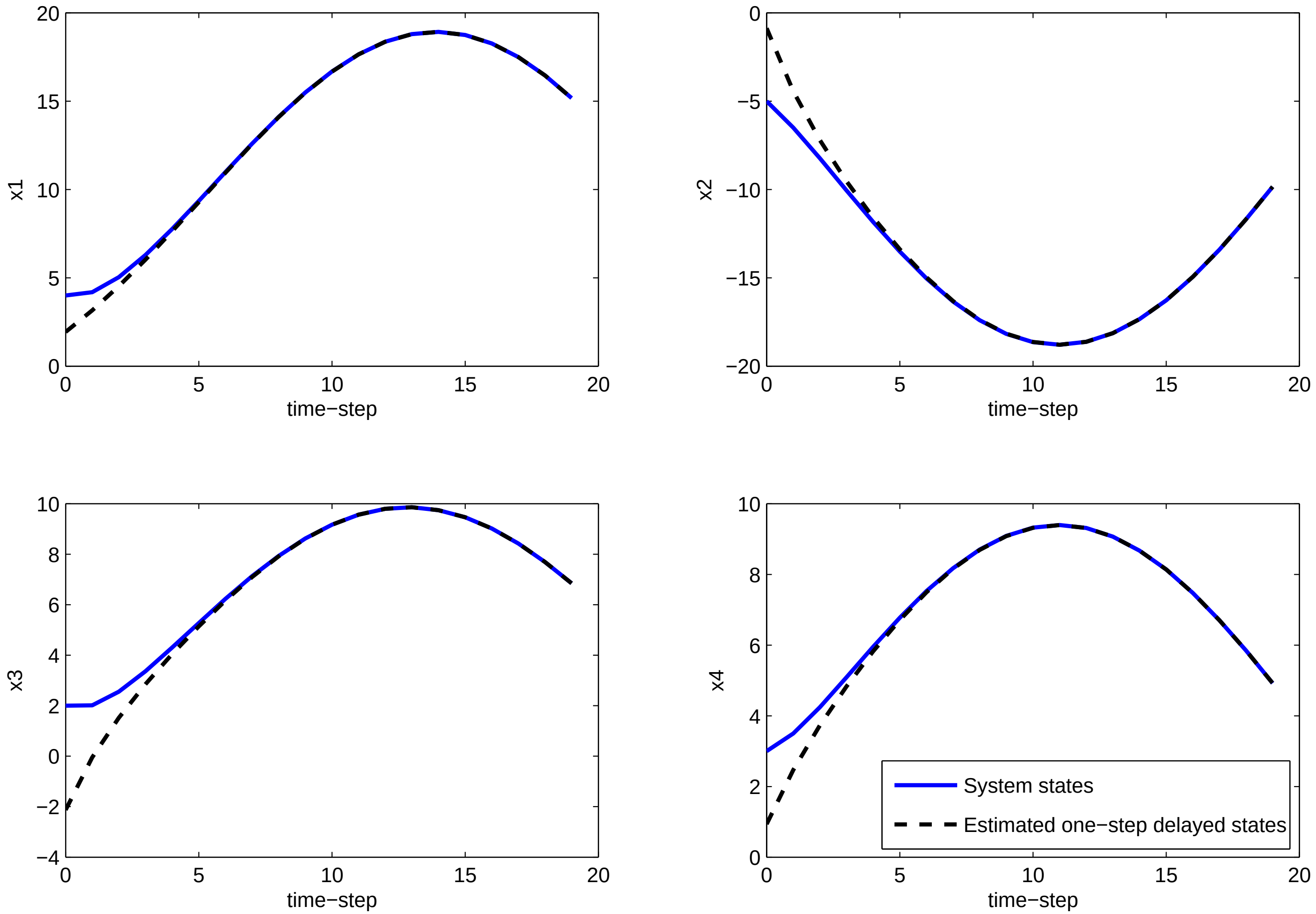
- Final observer given by

$$z_{k+1} = \frac{1}{2} z_k + F Y_{k:k+1}$$
$$\psi_k = z_k + G Y_{k:k+1}$$

- Estimate of original state given by

$$\hat{x}_k = \mathcal{T}^{-1} \begin{bmatrix} \mathcal{P} Y_{k:k+1} \\ \psi_k \end{bmatrix}$$

Example (7)



x1
time−step
x2
time−step
x3
time−step
x4
time−step
System states
Estimated one−step delayed states

Summary

- Provided a design procedure for delayed observers for linear systems with unknown inputs
  - Focused on reduced-order observers, allowing full-order observers as special case
- System inversion is necessary in order to construct an observer
  - Characterized the minimum and maximum delays required for state estimation
- Provided a parametrization of the observer gain to perform state and input decoupling
  - Remaining freedom used to ensure stability