Designing Stable Inverters and State Observers for Switched Linear Systems with Unknown Inputs

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Abstract— **We present a method for estimating the inputs and states in discrete-time switched linear systems with unknown inputs. We first investigate the problem of system invertibility, which reconstructs the unknown inputs based on knowledge of the output of the system, the switching sequence, and the initial system state. We then relax the assumption on the knowledge of the initial system state, and construct observers that asymptotically estimate the state. Our design, which considers a general class of switched linear observers that switch modes based on the known (but arbitrary) switching sequence, shows that system invertibility is necessary in order to construct state observers. Furthermore, some portion of the observer gain must be used to recover the unknown inputs, and the remaining freedom must be used to ensure stability. The state of the observer is then used to asymptotically estimate the unknown inputs (i.e., it forms the dynamic portion of a stable inverter for the given switched system).**

I. Introduction

In practice, it is often the case that a dynamic system can be modeled as having unknown inputs. The problem of estimating the unknown inputs in such systems based on the output of the system and the initial system state has been investigated extensively over the past few decades under the moniker of *system inversion* [7], [6]. These investigations have revealed that delayed (or differentiated) outputs will generally be necessary in order to reconstruct the unknown inputs. The problem of estimating the state in such systems has also received considerable attention over the past few decades [9], [8]. For linear time-invariant systems, it has been shown that the problem of state estimation is equivalent to that of stable system inversion, and so, delays will also generally be required in order to estimate the state [3], [8].

Recently, various researchers have investigated ways of observing the state of switched linear systems (e.g., see [1]). However, the concept of state observers for switched systems with unknown inputs has received only limited attention. In particular, Daafouz and Millerioux studied observer design for switched systems with unknown inputs in [5], but did not make use of delayed measurements.

In this paper, we study the problems of system inversion and state estimation in discrete-time switched linear systems with unknown inputs. We start by assuming knowledge of the initial system state, and provide necessary and sufficient conditions for the system to be invertible. We then relax the assumption on the knowledge of the initial system state, and design observers that asymptotically estimate the state. Our approach generalizes recent work on observers for linear time-invariant systems with unknown inputs [8], [9]. The main challenges in constructing these observers lie in i) decoupling the unknown inputs from the estimation error, and ii) ensuring that the estimation error decreases asymptotically to zero, regardless of the switching sequence. We show that system invertibility is necessary and sufficient to address the first challenge, and we present a parameterization of the observer gain that can be used to address the issue of stability. We then use the state observer to asymptotically estimate the unknown inputs, which produces a stable system inverse. The resulting inverse and state estimator can be used in applications such as fault-diagnosis and robust observer design for uncertain systems.

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II. Preliminaries

Consider a discrete-time switched linear system $\mathcal{S}$ of the form

$$\begin{aligned} x_{k+1} &= A_{\sigma[k]}x_k + B_{\sigma[k]}u_k \\ y_k &= C_{\sigma[k]}x_k + D_{\sigma[k]}u_k \end{aligned} \quad , \tag{1}$$

with state vector $x \in \mathbb{R}^n$, unknown input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and system matrices (A_i, B_i, C_i, D_i), $i \in \Omega = \{1, 2, \ldots, N\}$, of appropriate dimensions. Note that known inputs can be handled by a straightforward modification of our approach, and so we omit them in the above expression. The switching sequence is given by $\sigma : \mathbb{N} \rightarrow \Omega$. The dimension of the unknown input vector is taken to be a constant (with value m). However, it may be the case that each subsystem is affected by only a subset of these inputs. To handle this case, we will define $m_i = \text{rank}\left[\begin{smallmatrix} B_i \\ D_i \end{smallmatrix}\right]$, $i \in \Omega$, and we will assume without loss of generality that the first m_i columns of $\left[\begin{smallmatrix} B_i \\ D_i \end{smallmatrix}\right]$ are linearly independent, and the remaining columns are zero. In other words, we can write $\left[\begin{smallmatrix} B_i \\ D_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} B_{i,1} & 0 \\ D_{i,1} & 0 \end{smallmatrix}\right]$, where $B_{i,1}$ and $D_{i,1}$ each have m_i columns. This assumption can always be enforced by an appropriate (possibly time-varying) transformation of the unknown inputs. This means that the input to subsystem i consists only of the first m_i components of the input vector u.

We assume that the switching sequence is arbitrary, but known to our observer. For any given integer $\alpha \geq 0$, the switching path starting at time-step k and ending at time-step $k+\alpha$ will be denoted by $\sigma[k:k+\alpha]$. The response of system (1) over $\alpha+1$ time-steps is given by

$$Y_{k:k+\alpha} = \Theta_{\sigma[k:k+\alpha]} x_k + M_{\sigma[k:k+\alpha]} U_{k:k+\alpha} \ , \qquad (2)$$

where

$$Y_{k:k+\alpha} \equiv \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+\alpha} \end{bmatrix}, \ U_{k:k+\alpha} \equiv \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+\alpha} \end{bmatrix} \ ,$$

and the matrices $\Theta_{\sigma[k:k+\alpha]}$ and $M_{\sigma[k:k+\alpha]}$ can be defined recursively (for $\alpha \geq 0$) as

$$M_{\sigma[k:k]} = D_{\sigma[k]}, \ \Theta_{\sigma[k:k]} = C_{\sigma[k]} \ ,$$

$$\Theta_{\sigma[k:k+\alpha]} = \begin{bmatrix} \Theta_{\sigma[k:k+\alpha-1]} \\ C_{\sigma[k+\alpha]} \prod_{j=0}^{\alpha-1} A_{\sigma[k+j]} \end{bmatrix} \ , \qquad (3)$$

$$M_{\sigma[k:k+\alpha]} = \begin{bmatrix} D_{\sigma[k]} & 0 \\ \Theta_{\sigma[k+1:k+\alpha]} B_{\sigma[k]} & M_{\sigma[k+1:k+\alpha]} \end{bmatrix} \qquad (4)$$

$$= \begin{bmatrix} M_{\sigma[k:k+\alpha-1]} & 0 \\ C_{\sigma[k+\alpha]} \zeta_{\sigma[k:k+\alpha-1]} & D_{\sigma[k+\alpha]} \end{bmatrix} \ , \qquad (5)$$

where

$$\zeta_{\sigma[k:k+\alpha-1]} \equiv \begin{bmatrix} \prod_{j=1}^{\alpha-1} A_{\sigma[k+j]} B_{\sigma[k]} & \cdots & B_{\sigma[k+\alpha-1]} \end{bmatrix} \ .$$

III. Invertibility of Switched Systems

Definition 1: We say that the system in (1) is *invertible with delay* α if, at each time-step $k+\alpha$ ($k = 0, 1, \ldots$), it is possible to uniquely recover the first $m_{\sigma[k]}$ components of the unknown input u_k from the output of the system $Y_{k:k+\alpha}$, the switching sequence $\sigma[k:k+\alpha]$, and knowledge of the value of the state x_0. □

In the above definition, we are only interested in recovering the first $m_{\sigma[k]}$ components of the input u_k because of our assumption that the subsystem selected by $\sigma[k]$ is only affected by those components (from Section II). Furthermore, the above definition incorporates knowledge of the state x_0 in order to stay consistent with the notion of invertibility for non-switched systems (e.g., see [7]). The following theorem provides a characterization of the invertibility of the system (1).

Theorem 1: The system in (1) is invertible with delay α if and only if

$$\text{rank}\left[M_{\sigma[k:k+\alpha]}\right] - \text{rank}\left[M_{\sigma[k+1:k+\alpha]}\right] = m_{\sigma[k]} \qquad (6)$$

for all switching paths $\sigma[k:k+\alpha]$, where $m_{\sigma[k]} = \text{rank}\begin{bmatrix} B_{\sigma[k]} \\ D_{\sigma[k]} \end{bmatrix}$. □

By comparing (6) to the definition of $M_{\sigma[k:k+\alpha]}$ in (4), we note that equation (6) implies that the first $m_{\sigma[k]}$ columns of $M_{\sigma[k:k+\alpha]}$ must be linearly independent of each other, and of the remaining columns in $M_{\sigma[k:k+\alpha]}$. This interpretation will be used in the proof of the theorem.

Proof: We start by proving necessity. Suppose condition (6) does not hold for some switching path $\sigma[k:k+\alpha]$. Then for this switching path, there exists at least one column within the first $m_{\sigma[k]}$ columns of $M_{\sigma[k:k+\alpha]}$ that can be written as a linear combination of other columns in $M_{\sigma[k:k+\alpha]}$. Thus there exists a sequence of inputs $U_{k:k+\alpha}$ with at least one of the first $m_{\sigma[k]}$ components of u_k being nonzero such that $M_{\sigma[k:k+\alpha]} U_{k:k+\alpha} = 0$, and this is indistinguishable from the case where $U_{k:k+\alpha} = 0$. Therefore, it is not possible to uniquely determine the input by looking at the output of the system over $\alpha+1$ time-steps (given by equation (2)), which concludes the proof of necessity.

For sufficiency, suppose (6) holds for all switching sequences $\sigma[k:k+\alpha]$. Then for each such sequence, there exists a matrix $R_{\sigma[k:k+\alpha]}$ such that

$$R_{\sigma[k:k+\alpha]} M_{\sigma[k:k+\alpha]} = \begin{bmatrix} I_{m_{\sigma[k]}} & 0 & \cdots & 0 \end{bmatrix} \ . \qquad (7)$$

Left-multiplying (2) by $R_{\sigma[k:k+\alpha]}$ and rearranging, we obtain

$$\begin{bmatrix} I_{m_{\sigma[k]}} & 0 \end{bmatrix} u_k = -R_{\sigma[k:k+\alpha]} \Theta_{\sigma[k:k+\alpha]} x_k + R_{\sigma[k:k+\alpha]} Y_{k:k+\alpha} \ . \qquad (8)$$

Substituting equation (8) into (1), we obtain

$$x_{k+1} = (A_{\sigma[k]} - B_{\sigma[k],1} R_{\sigma[k:k+\alpha]} \Theta_{\sigma[k:k+\alpha]}) x_k + B_{\sigma[k],1} R_{\sigma[k:k+\alpha]} Y_{k:k+\alpha} \ , \qquad (9)$$

where $B_{\sigma[k],1}$ is the first $m_{\sigma[k]}$ columns of $B_{\sigma[k]}$. Since x_0 is known, we can recover the first $m_{\sigma[0]}$ components of u_0 uniquely from equation (8) by setting $k = 0$. We then use (9) to obtain the value of x_1, and the process can be repeated to obtain the first $m_{\sigma[k]}$ components of u_k for all k. Equations (9) and (8) together form the inverse system. ■

It is easy to show that the left side of (6) is a non-decreasing function of α. Thus, when investigating the invertibility of system (1), one can start with $\alpha = 0$ and increase α until a value is found that satisfies (6). In the single system case (i.e., $\Omega = \{1\}$), it is known that if the system is invertible, the upper bound on the delay in (6) will not exceed $\alpha = n - q + 1$, where q is the dimension of the nullspace of D [10]. However, it is not obvious how to extend this result to the switched case.

We now comment briefly on the complexity of checking condition (6) for a given α. At first glance, it appears that checking invertibility with delay α would require that we evaluate condition (6) for $N^{\alpha+1}$ different values of $M_{\sigma[k:k+\alpha]}$ (corresponding to the $N^{\alpha+1}$ different choices of $\sigma[k:k+\alpha]$). However, note that if condition (6) is satisfied for some particular switching sequence $\bar{\sigma}[k:k+\bar{\alpha}]$, where $\bar{\alpha} < \alpha$, then the system will also be invertible for any switching sequence $\sigma[k:k+\alpha]$ where $\sigma[k:k+\bar{\alpha}] = \bar{\sigma}[k:k+\bar{\alpha}]$. This can be easily seen by examining the structure of the matrix $M_{\sigma[k:k+\alpha]}$ in equation (5). Specifically, if the first $m_{\bar{\sigma}[k]}$ columns of $M_{\bar{\sigma}[k:k+\bar{\alpha}]}$ are linearly independent of each other and of the remaining columns in $M_{\bar{\sigma}[k:k+\bar{\alpha}]}$, then for any switching sequence $\sigma[k:k+\alpha]$ where $\sigma[k:k+\bar{\alpha}] = \bar{\sigma}[k:k+\bar{\alpha}]$, the matrix $M_{\sigma[k:k+\alpha]}$ will have $M_{\bar{\sigma}[k:k+\bar{\alpha}]}$ as a top-left subblock (from (5)). Since the top-right subblock in $M_{\sigma[k:k+\alpha]}$ is simply the zero matrix, the first $m_{\sigma[k]}$ columns of $M_{\sigma[k:k+\alpha]}$ will still be linearly independent of each other and of the remaining columns in the matrix. Intuitively, if we can reconstruct the input from the output $Y_{k:k+\bar{\alpha}}$ when the switching sequence is $\bar{\sigma}[k:k+\bar{\alpha}]$, then we can certainly determine the input from $Y_{k:k+\alpha}$ ($\alpha > \bar{\alpha}$) regardless of the switching sequence from time-step $k+\bar{\alpha}+1$ to time-step $k+\alpha$. Therefore, when checking invertibility with delay α, we do not need to consider switching sequences for which the system is invertible with delay less than α. The same reasoning also applies to the matrices $R_{\sigma[k:k+\alpha]}$ in (7). This observation can potentially reduce the number of matrices that have to be computed in order to invert the system.

Example 1: Consider a switched linear system of the form (1) that switches between two subsystems (i.e., $\Omega = \{1, 2\}$) given by

$$A_1 = \begin{bmatrix} 0.59 & -1.08 & 0.14 & 2.1 \\ 0.30 & -0.60 & 0.10 & 1.4 \\ 0.10 & -0.30 & 0.20 & 0.7 \\ 0.10 & -0.30 & -0.10 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 \\ -1 \\ -0.5 \\ -0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.7 & -0.3 & -0.75 & 0.1 \\ 0.1 & 0.4 & -0.60 & 0 \\ 0 & 0.1 & 0.05 & -0.1 \\ 0 & 0.1 & -0.35 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4.4 \\ 3.2 \\ 2 \\ 1.2 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We wish to determine if this system is invertible. Since $M_{\sigma[k:k]} \equiv D_{\sigma[k]} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for any $\sigma[k] \in \Omega$, the system is not invertible with delay 0. To investigate invertibility with delay $\alpha = 1$, we consider the matrix

$$M_{\sigma[k:k+1]} \equiv \begin{bmatrix} D_{\sigma[k]} & 0 \\ C_{\sigma[k+1]}B_{\sigma[k]} & D_{\sigma[k+1]} \end{bmatrix}.$$

Since $D_{\sigma[k]}$ is zero for all k, we have $\text{rank}\left[M_{\sigma[k:k+1]}\right] - \text{rank}\left[M_{\sigma[k+1:k+1]}\right] = \text{rank}[C_{\sigma[k+1]}B_{\sigma[k]}]$. If $\sigma[k] = \sigma[k+1]$, we have $\text{rank}\left[C_{\sigma[k+1]}B_{\sigma[k]}\right] = 1$, and so the inversion condition given by (6) is satisfied for these switching sequences. However, if $\sigma[k] \neq \sigma[k+1]$, we have $C_{\sigma[k+1]}B_{\sigma[k]} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, and so the system is not invertible with delay $\alpha = 1$ (i.e., the input cannot be uniquely identified for switching sequences where $\sigma[k] \neq \sigma[k+1]$).

Next, we consider switching sequences of length 3 (i.e., a delay of $\alpha = 2$). Since the system is invertible for any switching sequence where $\sigma[k] = \sigma[k+1]$ (regardless of the value of $\sigma[k+2]$), we only have to investigate invertibility for the sequences $\sigma[k:k+2] \in \{\{1,2,1\},\{1,2,2\},\{2,1,1\},\{2,1,2\}\}$. To check invertibility for these sequences, we need to consider the matrix

$$M_{\sigma[k:k+2]} \equiv \begin{bmatrix} D_{\sigma[k]} & 0 & 0 \\ C_{\sigma[k+1]}B_{\sigma[k]} & D_{\sigma[k+1]} & 0 \\ C_{\sigma[k+2]}A_{\sigma[k+1]}B_{\sigma[k]} & C_{\sigma[k+2]}B_{\sigma[k+1]} & D_{\sigma[k+2]} \end{bmatrix}.$$

For the four switching sequences listed above, we find that $\text{rank}[M_{\sigma[k:k+2]}] - \text{rank}[M_{\sigma[k+1:k+2]}] = 1$, and so the system is invertible for all switching sequences of length 3 (i.e., the system is invertible with delay $\alpha = 2$).

Note that the inverse system (given by (9) and (8)) will only require six different values of the matrices $R_{\sigma[k:k+2]}$, corresponding to switching sequences $\{\{1,1,*\},\{1,2,1\},\{1,2,2\},\{2,1,1\},\{2,1,2\},\{2,2,*\}\}$, where $*$ represents an arbitrary switching value. For example, for the switching sequence $\sigma[k:k+2] = \{1,1,*\}$, we choose $R_{\{1,1,*\}} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$, and this satisfies the equation $R_{\{1,1,*\}}M_{\{1,1,*\}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. The $R_{\sigma[k:k+2]}$ matrices for the other switching sequences can be found in a similar manner. □

Examining the inverse system specified by equations (9) and (8), we see that the dynamic portion of the inverse system (equation (9)) simply reconstructs the state x_k, which is then used to reproduce the input $\begin{bmatrix} I_{m_{\sigma[k]}} & 0 \end{bmatrix} u_k$ in equation (8). Furthermore, since we assumed knowledge of x_0, equation (9) will produce the exact value of the state at each time-step. In the next section, we will drop the assumption that the initial state x_0 is known, and try to construct a state observer to asymptotically estimate the state of the system.

IV. State Estimation

To estimate the system states, we consider an observer of the form

$$\hat{x}_{k+1} = E_{\sigma[k:k+\alpha]}\hat{x}_k + K_{\sigma[k:k+\alpha]}Y_{k:k+\alpha} \tag{10}$$

where matrices $E_{\sigma[k:k+\alpha]}$ and $K_{\sigma[k:k+\alpha]}$ are chosen such that $\hat{x}_k \to x_k$ as $k \to \infty$. Note that the form of the observer is reminiscent of the standard Luenberger observer, with the obvious extension to a switched system, together with delays in the observer. Also note that the dynamic portion of the system inverse (equation (9)) shares a similar form as well. Using (2), the observer error is given by

$$\begin{aligned} e_{k+1} &\equiv \hat{x}_{k+1} - x_{k+1} \\ &= E_{\sigma[k:k+\alpha]}\hat{x}_k + K_{\sigma[k:k+\alpha]}Y_{k:k+\alpha} - A_{\sigma[k]}x_k \\ &\quad - B_{\sigma[k]}u_k \\ &= E_{\sigma[k:k+\alpha]}e_k \\ &\quad + \left(E_{\sigma[k:k+\alpha]} - A_{\sigma[k]} + K_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]}\right)x_k \\ &\quad + \left(K_{\sigma[k:k+\alpha]}M_{\sigma[k:k+\alpha]} - \begin{bmatrix} B_{\sigma[k]} & 0 \end{bmatrix}\right)U_{k:k+\alpha}. \end{aligned} \tag{11}$$

Since the unknown inputs and states can have arbitrary values, they must be decoupled from the estimation error. This means that the following two conditions must hold for all time-steps k (i.e., for all switching paths $\sigma[k:k+\alpha]$):

$$K_{\sigma[k:k+\alpha]}M_{\sigma[k:k+\alpha]} = \begin{bmatrix} B_{\sigma[k]} & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

$$E_{\sigma[k:k+\alpha]} = A_{\sigma[k]} - K_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]}. \tag{13}$$

Furthermore, the matrix $E_{\sigma[k:k+\alpha]}$ must be chosen to ensure that the estimation error decreases to zero, regardless of the switching sequence. We refer to equation (12) as the input decoupling condition, and we study it next. After we satisfy the input decoupling condition, we will study the problem of ensuring stability in (13).

A. Input Decoupling

The solvability of condition (12) is given by the following theorem. The proof of the theorem is given in the Appendix.

Theorem 2: There exists a matrix $K_{\sigma[k:k+\alpha]}$ satisfying (12) if and only if

$$\text{rank}\left[M_{\sigma[k:k+\alpha]}\right] - \text{rank}\left[M_{\sigma[k+1:k+\alpha]}\right] = \text{rank}\begin{bmatrix} B_{\sigma[k]} \\ D_{\sigma[k]} \end{bmatrix}$$

for all switching paths $\sigma[k:k+\alpha]$ (i.e., the system must be invertible with a delay of α). □

Remark 1: In [5], the authors consider the problem of estimating the states in the switched system (1) with $D_{\sigma[k]} = 0$ for all k, and construct an observer that estimates x_{k+1} using the output y_{k+1}. This corresponds to choosing $\alpha = 1$ in our setup. Under these conditions, equation (6) becomes $\text{rank}[C_{\sigma[k+1]}B_{\sigma[k]}] = \text{rank}[B_{\sigma[k]}]$ for all k. However, the condition provided in [5] to decouple the inputs is $\text{rank}[C_{\sigma[k]}B_{\sigma[k]}] = \text{rank}[B_{\sigma[k]}]$ for all k. This condition is necessary (i.e., if $\sigma[k+1] = \sigma[k]$), but not sufficient to guarantee the input decoupling condition, contrary to what is stated in [5] (e.g., see Example 1). However, this error does not affect the system considered in their example, because it has a constant C matrix. □

Theorem 2 implies that the first $m_{\sigma[k]}$ columns of $M_{\sigma[k:k+\alpha]}$ must be linearly independent, and cannot be written as a linear combination of other columns in $M_{\sigma[k:k+\alpha]}$. Let $\mathcal{N}_{\sigma[k:k+\alpha]}$ be a matrix whose rows form a basis for the left nullspace of the last $(\alpha+1)m - m_{\sigma[k]}$ columns of $M_{\sigma[k:k+\alpha]}$. In particular, we can assume without loss of generality that $\mathcal{N}_{\sigma[k:k+\alpha]}$ satisfies

$$\mathcal{N}_{\sigma[k:k+\alpha]}M_{\sigma[k:k+\alpha]} = \begin{bmatrix} 0 & 0 \\ I_{m_{\sigma[k]}} & 0 \end{bmatrix} \quad . \tag{14}$$

Note that there exists an $\mathcal{N}_{\sigma[k:k+\alpha]}$ satisfying the above equation if and only if (6) is satisfied. Just as in Section III, we do not necessarily have to calculate $N^{\alpha+1}$ different values of $\mathcal{N}_{\sigma[k:k+\alpha]}$. In particular, if $M_{\bar{\sigma}[k:k+\bar{\alpha}]}$ satisfies (6) for some $\bar{\alpha} < \alpha$ and some switching sequence $\bar{\sigma}[k:k+\bar{\alpha}]$, then for that particular switching sequence, we can calculate a matrix $\bar{\mathcal{N}}_{\bar{\sigma}[k:k+\bar{\alpha}]}$ such that

$$\bar{\mathcal{N}}_{\bar{\sigma}[k:k+\bar{\alpha}]}M_{\bar{\sigma}[k:k+\bar{\alpha}]} = \begin{bmatrix} 0 & 0 \\ I_{m_{\bar{\sigma}[k]}} & 0 \end{bmatrix} \quad .$$

Then for all extensions of this switching sequence (i.e., for all $\sigma[k:k+\alpha]$ such that $\sigma[k:k+\bar{\alpha}] = \bar{\sigma}[k:k+\bar{\alpha}]$), we can choose $\mathcal{N}_{\sigma[k:k+\alpha]} = \begin{bmatrix} \bar{\mathcal{N}}_{\bar{\sigma}[k:k+\bar{\alpha}]} & 0 \end{bmatrix}$, where the zero matrix has appropriate dimensions.

Based on the property of $\mathcal{N}_{\sigma[k:k+\alpha]}$ in equation (14), we see that (12) will be satisfied if $K_{\sigma[k:k+\alpha]}$ is of the form

$$K_{\sigma[k:k+\alpha]} = \widehat{K}_{\sigma[k:k+\alpha]}\mathcal{N}_{\sigma[k:k+\alpha]}$$

for some $\widehat{K}_{\sigma[k:k+\alpha]} \equiv \begin{bmatrix} \mathcal{L}_{\sigma[k:k+\alpha]} & \widetilde{K}_{\sigma[k:k+\alpha]} \end{bmatrix}$, where $\widetilde{K}_{\sigma[k:k+\alpha]}$ has $m_{\sigma[k]}$ columns. Equation (12) then becomes

$$\begin{bmatrix} \mathcal{L}_{\sigma[k:k+\alpha]} & \widetilde{K}_{\sigma[k:k+\alpha]} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_{m_{\sigma[k]}} & 0 \end{bmatrix} = \begin{bmatrix} B_{\sigma[k],1} & 0 \end{bmatrix} \quad ,$$

from which it is obvious that $\widetilde{K}_{\sigma[k:k+\alpha]} = B_{\sigma[k],1}$ and $\mathcal{L}_{\sigma[k:k+\alpha]}$ is a free matrix (recall that $B_{\sigma[k]} \equiv \begin{bmatrix} B_{\sigma[k],1} & 0 \end{bmatrix}$).

Returning to the condition in equation (13), we have

$$\begin{aligned} E_{\sigma[k:k+\alpha]} &= A_{\sigma[k]} - K_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]} \\ &= A_{\sigma[k]} \\ &\quad - \begin{bmatrix} \mathcal{L}_{\sigma[k:k+\alpha]} & B_{\sigma[k],1} \end{bmatrix} \mathcal{N}_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]} \quad . \end{aligned}$$

Defining

$$\begin{bmatrix} \mathcal{C}_{\sigma[k:k+\alpha]} \\ \Phi_{\sigma[k:k+\alpha]} \end{bmatrix} \equiv \mathcal{N}_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]} \quad , \tag{15}$$

where $\Phi_{\sigma[k:k+\alpha]}$ has $m_{\sigma[k]}$ rows, we come to the equation

$$\begin{aligned} E_{\sigma[k:k+\alpha]} = &\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}\right) \\ &- \mathcal{L}_{\sigma[k:k+\alpha]}\mathcal{C}_{\sigma[k:k+\alpha]} \quad . \end{aligned}$$

Substituting the above expression into the formula for the error given in (11), we get

$$\begin{aligned} e_{k+1} = \big(&\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}\right) \\ &- \mathcal{L}_{\sigma[k:k+\alpha]}\mathcal{C}_{\sigma[k:k+\alpha]}\big)\, e_k \quad . \end{aligned} \tag{16}$$

To analyze this equation, let T represent the number of unique pairs $\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}, \mathcal{C}_{\sigma[k:k+\alpha]}\right)$. Define the surjective map $\mu : \Omega^{\alpha+1} \rightarrow \Psi = \{1, \ldots, T\}$ that assigns to each switching path of length $\alpha+1$ a number from the set Ψ identifying the pair $\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}, \mathcal{C}_{\sigma[k:k+\alpha]}\right)$ corresponding to that path. For convenience, we will write $\mu[\sigma[k:k+\alpha]]$ as $\mu[k]$. Note that μ is not necessarily injective (one-to-one), since different switching paths could produce the same pair $\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}, \mathcal{C}_{\sigma[k:k+\alpha]}\right)$. Thus, for each switching path $\sigma[k:k+\alpha]$, we can define the set of matrices

$$\begin{aligned} \mathcal{A}_{\mu[k]} \equiv A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+\alpha]}, \quad \mathcal{C}_{\mu[k]} \equiv \mathcal{C}_{\sigma[k:k+\alpha]} \quad , \\ \mathcal{L}_{\mu[k]} \equiv \mathcal{L}_{\sigma[k:k+\alpha]} \quad . \end{aligned} \tag{17}$$

Therefore, the expression for the error from (16) can be written as

$$e_{k+1} = \left(\mathcal{A}_{\mu[k]} - \mathcal{L}_{\mu[k]}\mathcal{C}_{\mu[k]}\right) e_k \quad . \tag{18}$$

Note that (18) is a switched linear system under *constrained* switching. To see this, note that since $\mu[k] \equiv \mu[\sigma[k:k+\alpha]]$ and $\mu[k+1] \equiv \mu[\sigma[k+1:k+\alpha+1]]$, the switching signals $\mu[k]$ and $\mu[k+1]$ are both functions of the switching path $\sigma[k+1:k+\alpha]$, and so the values of $\mu[k]$ and $\mu[k+1]$ will be related accordingly.

Example 2: We return to the system introduced in Example 1. From the analysis in that example, we noted that the minimum delay to invert the system is $\alpha = 2$, and so the state observer will also require a delay of two time-steps (from Theorem 2). Furthermore, we also noted in Example 1 that for switching sequences that satisfy $\sigma[k] = \sigma[k+1]$, the system is invertible with a delay of one time-step. Therefore, we only need to calculate six different values for the $N_{\sigma[k:k+2]}$ matrices in (14), corresponding to the switching sequences $\{\{1,1,*\},\{1,2,1\},\{1,2,2\},\{2,1,1\},\{2,1,2\},\{2,2,*\}\}$,

TABLE I
CHARACTERIZATION OF SWITCHING SIGNAL μ.

$\sigma[k:k+2]$	$\mu[k]$	$\mu[k+1]$
$\{1,1,*\}$	1	$1,2,3$
$\{1,2,1\}$	2	$4,5$
$\{1,2,2\}$	3	6
$\{2,1,1\}$	4	1
$\{2,1,2\}$	5	$2,3$
$\{2,2,*\}$	6	$4,5,6$

where $*$ represents an arbitrary switching value. The explicit values of these matrices are omitted in the interest of space. In this example, each of the above switching sequences corresponds to a different pair $\left(A_{\sigma[k]} - B_{\sigma[k],1}\Phi_{\sigma[k:k+2]}, \mathcal{C}_{\sigma[k:k+2]}\right)$ in (16). The map $\mu[\sigma[k:k+2]]$ which assigns a unique identifier to each of these different pairs is shown in the second column of Table I. Furthermore, for a given $\mu[k]$, $\mu[k+1]$ can only take on a restricted set of values, which are shown in the third column of Table I. For example, consider the second row of Table I. If $\mu[k]=2$, then $\sigma[k:k+2]=\{1,2,1\}$. Therefore, the possible values for $\sigma[k+1:k+3]$ are given by $\{\{2,1,1\},\{2,1,2\}\}$, which correspond to the identifiers $\{4,5\}$. The other entries of Table I are generated by following the same reasoning. □

B. Stability

Consider the expression for the error in (18). We require this error to asymptotically decay to zero, regardless of the switching signal μ. If this property is satisfied, the error is termed to be *globally uniformly asymptotically stable* (GUAS). Unfortunately, there are no known necessary and sufficient conditions to analyze the stability of (18) under general conditions. However, there are various sufficient conditions that will guarantee stability [4], and we can attempt to use any of these conditions to choose the matrices $\mathcal{L}_{\mu[k]}$ in order to make the error GUAS. For completeness, we use the method in [2] as an example. This method uses the following theorem to characterize the stability of switched systems of the form (18).

Theorem 3: If there exists a set of symmetric matrices S_i and some matrices R_i and W_i such that

$$\begin{bmatrix} W_i + W_i^T - S_i & W_i^T\mathcal{A}_i - R_i^T\mathcal{C}_i \\ \mathcal{A}_i^T W_i - \mathcal{C}_i^T R_i & S_j \end{bmatrix} > 0 \ , \qquad (19)$$

for all (i,j) such that $i=\mu[k]$ and $j=\mu[k+1]$, then the switched system (18) will be GUAS and the resulting gains are given by $\mathcal{L}_i = \left(R_i W_i^{-1}\right)^T$. □

When applying the above theorem, note that for a given $i \in \Psi$, the parameter j can only take on a restricted set of values, due to the constrained switching sequence μ. As discussed in [2], (19) is a Linear Matrix Inequality (LMI), which can be efficiently solved by using convex optimization techniques.

Regardless of the method used, once we find a set of $\mathcal{L}_i$'s, $i \in \Psi$, such that the error dynamics given in (18) are GUAS,

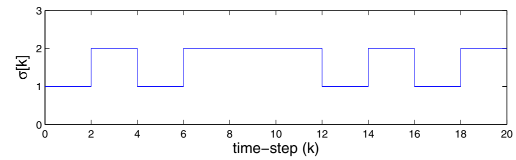

Fig. 1. Switching sequence $\sigma[k]$.

we can obtain the observer gains in (10) as

$$E_{\sigma[k:k+\alpha]} = \mathcal{A}_{\mu[k]} - \mathcal{L}_{\mu[k]}\mathcal{C}_{\mu[k]} \ , \qquad (20)$$

$$K_{\sigma[k:k+\alpha]} = \begin{bmatrix}\mathcal{L}_{\mu[k]} & B_{\sigma[k],1}\end{bmatrix}\mathcal{N}_{\sigma[k:k+\alpha]} \ , \qquad (21)$$

and this will ensure that the state observer in (10) will asymptotically estimate the state, regardless of the initial state and the values of the unknown inputs.

Example 3: We return to the system investigated in Examples 1 and 2. Recall that the system is invertible with a delay of $\alpha=2$, and the error dynamics switch between a set of six different realizations (i.e., $\mu[k] \in \Psi = \{1,\ldots,6\}$). To design a stable state estimator, we need to find a set of matrices $\mathcal{L}_i$, $i \in \Psi$, so that the error dynamics given by (18) are stable for any admissible switching sequence $\mu[k]$. To apply Theorem 3, we need to solve the LMIs given in (19) for each possible value of $\mu[k]$ and $\mu[k+1]$. From Table I, we see that there will be twelve of these LMIs that have to be solved simultaneously (i.e., three LMIs corresponding to the case $\mu[k]=1$, two LMIs corresponding to the case $\mu[k]=2$, and so forth). Using the LMI toolbox in MATLAB, we find that the set of LMIs has a feasible solution, and obtain the corresponding values of $\mathcal{L}_i$, $i\in\Psi$. We then use these matrices to obtain the observer gains $E_{\sigma[k:k+2]}$ and $K_{\sigma[k:k+2]}$ from equations (20) and (21). We omit the explicit values of these matrices in the interest of space. The final observer is obtained from equation (10) as

$$\hat{x}_{k+1} = E_{\sigma[k:k+2]}\hat{x}_k + K_{\sigma[k:k+2]}Y_{k:k+2} \ . \qquad (22)$$

To test the observer, we initialize the original system (1) with a randomly chosen state ($\begin{bmatrix}-4 & -2 & -2 & -2\end{bmatrix}^T$), and apply the randomly chosen switching sequence $\sigma[k]$ shown in Fig. 1. For the unknown input, we use the randomly chosen signal

$$u_k = 0.1\left(\sin\frac{2\pi k}{40} + 0.9\cos\frac{2\pi k}{40}\right) + r_k \ ,$$

where $r_k = 1$ if $0 \le k \le 9$, and zero otherwise. The observer is initialized with zero initial state. The results of the simulation are shown in Fig. 2, and we see that the estimated state asymptotically catches up to the actual state. Note that the estimated state should technically be delayed by two time-steps (since the observer has a delay of $\alpha=2$), but we have shifted the estimate forward to allow better comparison. □

Now that we have constructed a state observer, we will apply it to the problem of stable system inversion.

V. STABLE INVERSION

Suppose that the system is invertible, and the techniques discussed in the last section produce an observer of the form

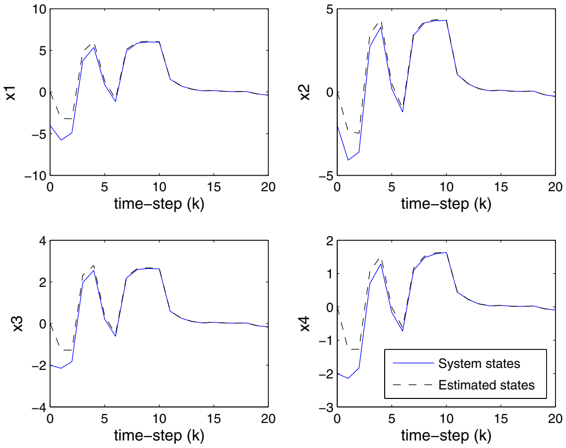

Fig. 2. Actual and estimated states.

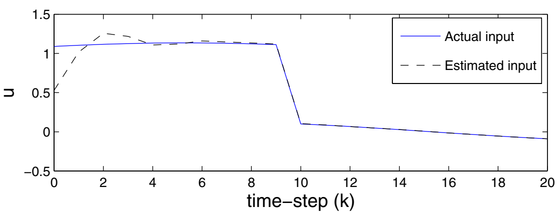

Fig. 3. Actual and estimated inputs.

(10) that asymptotically estimates the state. Using (8) as a guide, we can then write

$$\hat{u}_k = -R_{\sigma[k:k+\alpha]}\Theta_{\sigma[k:k+\alpha]}\hat{x}_k + R_{\sigma[k:k+\alpha]}Y_{k:k+\alpha} \quad , \tag{23}$$

where $\hat{x}_k$ is the estimated state from equation (10). Since $\hat{x}_k \to x_k$ as $k \to \infty$, we see from equations (23) and (8) that $\hat{u}_k \to \begin{bmatrix} I_{m_{\sigma[k]}} & 0 \end{bmatrix} u_k$ as $k \to \infty$. Note also that $R_{\sigma[k:k+\alpha]}$ can be any matrix that satisfies equation (7). In particular, by examining the definition of the matrix $\mathcal{N}_{\sigma[k:k+\alpha]}$ in (14), we see that we can simply choose $R_{\sigma[k:k+\alpha]}$ to be the last $m_{\sigma[k]}$ rows of $\mathcal{N}_{\sigma[k:k+\alpha]}$.

Example 4: In Example 3, we constructed the state observer in equation (22). To estimate the inputs, we choose the matrices $R_{\sigma[k:k+2]}$ as the last row of the matrices $\mathcal{N}_{\sigma[k:k+2]}$ (since $m_{\sigma[k]} = 1$ for all k). We then obtain an estimate of u_k from equation (23), with $\alpha = 2$. The convergence of $\hat{u}_k$ to the actual input u_k (which was specified in Example 3) is shown in Fig. 3. Once again, the estimated input should technically be delayed by $\alpha = 2$ time-steps, but we have shifted it forward to allow better comparison. □

VI. Conclusions and Future Work

We have studied the problem of estimating the states and inputs in discrete-time switched linear systems with unknown inputs. We started by constructing an inverse system to reconstruct the inputs under the assumption that the initial system state is known. We then removed our assumption that the initial state is known, and we studied how to build observers that asymptotically estimate the system state. We then used this observer as the dynamic portion of the system inverse, thereby obtaining asymptotic estimates of the unknown inputs.

There are some interesting directions for future research. For example, if the switching sequence is unknown, how does one simultaneously estimate the state, unknown inputs and switching sequence? It will also be interesting to study the robustness of our estimators and inverters to parametric uncertainties.

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Appendix

Proof: [Theorem 2] There exists a $K_{\sigma[k:k+\alpha]}$ satisfying (12) if and only if the row space of the matrix $\bar{B}_{\sigma[k]} \equiv \begin{bmatrix} B_{\sigma[k]} & 0 \end{bmatrix}$ is in the space spanned by the rows of $M_{\sigma[k:k+\alpha]}$. This is equivalent to the condition

$$\text{rank}\begin{bmatrix} M_{\sigma[k:k+\alpha]} \\ \bar{B}_{\sigma[k]} \end{bmatrix} = \text{rank}\begin{bmatrix} M_{\sigma[k:k+\alpha]} \end{bmatrix} \quad . \tag{24}$$

Using (4), we get

$$\begin{aligned} &\text{rank}\begin{bmatrix} M_{\sigma[k:k+\alpha]} \\ \bar{B}_{\sigma[k]} \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} D_{\sigma[k]} & 0 \\ \Theta_{\sigma[k+1:k+\alpha]}B_{\sigma[k]} & M_{\sigma[k+1:k+\alpha]} \\ B_{\sigma[k]} & 0 \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} B_{\sigma[k]} \\ D_{\sigma[k]} \end{bmatrix} + \text{rank}\begin{bmatrix} M_{\sigma[k+1:k+\alpha]} \end{bmatrix} \quad . \end{aligned}$$

Substituting this into (24) completes the proof. ■

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