Stability of Dynamical Systems on a Graph

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***Abstract*— We study the stability of large-scale discrete-time dynamical systems that are composed of interconnected subsystems. The stability of such systems is a function of both the dynamics and the interconnection topology. We investigate two notions of stability; the first is connective stability, where the overall system is stable in the sense of Lyapunov despite uncertainties and time-variations in the coupling strengths between subsystems. The second is the standard notion of asymptotic (Schur) stability of the overall system, assuming all interconnections are fixed at their nominal levels. We make connections to spectral graph theory, and specifically the spectra of signed adjacency matrices, to provide graph theoretic characterizations of the two kinds of stability for the case of homogeneous scalar subsystems. In the process, we derive bounds on the largest eigenvalue of signed adjacency matrices that are of independent interest.**

I. Introduction

Complex systems (such as the power grid, the Internet, biological systems, ecological systems, economical systems and social systems) are composed of smaller interconnected subsystems. The behavior of the overall system is governed by both the dynamics of the individual subsystems and the nature of the interconnections, and the interplay between the dynamics and topology of networked systems is an active area of research across multiple disciplines [1]–[4].

A topic of particular interest in the study of such systems is *stability*: can the interconnections combine with the individual dynamics to cause the state of the overall system to spiral out of control? Early work by May [5] studied random interaction structures in population dynamics, and showed that instability occurs once the coupling strength between subsystems increases past a critical value; see also [6], [7]. The design of network topologies and feedback patterns for stabilization of dynamical systems has also been investigated in the context of decentralized control [8], [9]. Recent work has investigated the sparsity patterns (and associated topological conditions) that are required for a matrix to represent a Hurwitz system [10].

In this paper, our goal is to study the stability of a network of discrete-time linear systems, and to give topological conditions for the overall system to be stable. Specifically, we consider two notions of stability. The first, known as *connective stability*, is a strong notion of stability where the state of the system asymptotically goes to zero despite uncertainties or time-variations in the coupling between the subsystems [8]. The second is the standard notion of asymptotic (Schur) stability, where the state of the system goes to zero for fixed (known and time-invariant) coupling. The former implies the latter, and thus sufficient conditions for connective stability are also sufficient for Schur stability. Conversely, necessary conditions for Schur stability are also necessary for connective stability. For the case of scalar subsystems, we provide graph-theoretic bounds for the self-dynamics and coupling strengths in order to obtain the two kinds of stability. As we will discuss, the two notions of stability are equivalent when all of the self-dynamics and coupling strengths are nonnegative, or when the sign patterns of the interconnections satisfy a certain notion of structural balance. The gap between connective and Schur stability becomes more pronounced when the signs of the interconnections are allowed to be different. To analyze this scenario, we make connections to spectral graph theory, and in particular, the spectral theory of signed adjacency matrices, to obtain bounds on the largest eigenvalue of the system dynamics matrix. The bounds that we obtain are of independent interest in the study of signed graphs.

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II. Notation and Terminology

The set of positive real numbers is denoted by $\mathbb{R}_+$. Element (i, j) in a matrix A will be denoted by a_{ij}. The norm of a matrix A will be denoted by $\|A\|$. For a symmetric $n \times n$ matrix A, we will arrange the eigenvalues as

$$\lambda_{max}(A) = \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) = \lambda_{min}(A).$$

The *spectral radius* of a square $n \times n$ matrix A is given by $\rho(A) \triangleq \max_{1\leq i\leq n} |\lambda_i(A)|$. For a matrix A, the notation $A \geq 0$ indicates that each element of the matrix is nonnegative, with analogous definitions for $A > 0$, $A < 0$ and $A \leq 0$. The notation $|A|$ indicates the matrix obtained by taking the absolute value of all elements in matrix A.

An $n\times n$ matrix A with nonpositive off-diagonal elements is called an *M-matrix* if it satisfies one of the following (equivalent) properties [1]:

1) There exists a vector $c \in \mathbb{R}^n_+$ such that $Ac > 0$.
2) The real part of each eigenvalue of A is positive.

A. Graph Theoretic Terminology

We use $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ to denote a graph where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices (or nodes) and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges; we assume that $(v_i, v_i) \notin \mathcal{E}$ for all $v_i \in \mathcal{V}$. The graph is said to be *undirected* if $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. The *neighbors* of vertex $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i(\mathcal{G}) = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The *degree* of vertex $v_i \in \mathcal{V}$ is given by $d_i(\mathcal{G}) \triangleq |\mathcal{N}_i(\mathcal{G})|$. When the context is clear, we will drop the dependence on $\mathcal{G}$ when referring to the neighborhood and degree of a given vertex. A *cycle* in the graph is a sequence of vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$ such that no vertex other than v_{i_1} appears more than once in the list and each pair of consecutive vertices in the sequence has an edge from the first vertex to the second.

A *subgraph* of $\mathcal{G}$ is a graph $\mathcal{H} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$, with $\bar{\mathcal{V}} \subseteq \mathcal{V}$ and $\bar{\mathcal{E}} \subseteq \mathcal{E} \cap (\bar{\mathcal{V}} \times \bar{\mathcal{V}})$; the subgraph is *induced* if equality holds in the latter expression. A *clique* in graph $\mathcal{G}$ is an induced subgraph $\mathcal{H} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ of $\mathcal{G}$ such that $\bar{\mathcal{E}}$ consists of edges between all pairs of different vertices in $\bar{\mathcal{V}}$. An undirected graph is a *star centered at* v_i if all vertices other than v_i have a single edge to v_i.

The *weighted adjacency matrix* of a graph $\mathcal{G}$ is a matrix $A \in \mathbb{R}^{n \times n}$, where element $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$, and nonzero otherwise. When the nonzero elements of a weighted adjacency matrix take values in the set $\{-1, 1\}$, it is said to be a *signed adjacency matrix*. When all nonzero elements are equal to 1, one obtains the classical (unsigned) adjacency matrix. Associated with each weighted adjacency matrix is a weighted graph, with weight a_{ij} on edge (v_j, v_i); a *signed graph* is a graph whose edges can take positive or negative values. A signed graph is called *structurally balanced* if the multiplication of the signs through each cycle in the graph is positive. Note that the diagonal elements of adjacency matrices are zero when there are no self-loops in the graph. The *spectrum* of a graph $\mathcal{G}$ is the set of eigenvalues of the adjacency matrix A. More information on graphs can be found in standard textbooks, such as [11], [12].

III. Connective Stability

Consider a discrete-time linear system $\mathcal{S}$ composed of N subsystems, where each subsystem is of the form

$$x_i(k+1) = A_{ii}x_i(k) + \sum_{\substack{j=1 \\ j \neq i}}^{N} e_{ij}A_{ij}x_j(k), \ \ 1 \leq i \leq N, \quad (1)$$

with state vector $x_i \in \mathbb{R}^{n_i}$ and dynamics matrix $A_{ii} \in \mathbb{R}^{n_i \times n_i}$. The term $A_{ij}x_j(k)$ represents the coupling from subsystem j to subsystem i. The quantity $e_{ij} \in [0, 1]$ represents an uncertain coupling strength which varies between 0 (no coupling) and 1 (maximum coupling). The state of system $\mathcal{S}$ is given by the vector $x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T$.

Definition 1 ([8]): The system $\mathcal{S}$ is connectively stable if it is stable in the sense of Lyapunov for all $e_{ij} \in [0, 1]$, $i, j \in \{1, 2, \ldots, N\}$. □

The above definition of connective stability implies that the system without any coupling (i.e., $e_{ij} = 0$ for all i, j) must also be stable. Therefore, a necessary condition for connective stability is that A_{ii} must be Schur for $i \in \{1, 2, \ldots, N\}$.

One can obtain a sufficient condition for connective stability by using the notion of vector Lyapunov functions [8], [13]. For the subsystems in (1), the coupling terms can be bounded using the inequality

$$\| \sum_{j=1}^{N} e_{ij}A_{ij}x_j(k) \|_2 \leq \sum_{j=1}^{N} e_{ij} \|A_{ij}\|_2 \|x_j\|_2. \quad (2)$$

Given that A_{ii} is stable, it is possible to find a Lyapunov function of the form

$$v_i(x_i) = (x_i^T H_i x_i)^{1/2}, \forall i \in \{1, 2, \ldots, N\} \quad (3)$$

where H is a positive definite matrix that satisfies

$$A_{ii}^T H_i A_{ii} - H_i = -G_i \quad (4)$$

for some positive definite matrix G_i. To deal with additive perturbations (such as coupling), one can define a *robustness bound* for each subsystem based on its Lyapunov function, characterizing the size of the perturbation that can be tolerated while still having the Lyapunov function decrease at each time-step.

Definition 2 ([13]): The robustness bound of the Lyapunov function (3) for each subsystem is given by

$$\xi_v(G_i) = \frac{\lambda_{min}(G_i)}{\lambda_{max}^{1/2}(H_i)\lambda_{max}^{1/2}(H_i - G_i) + \lambda_{max}(H_i)}. \quad (5)$$

□

The robustness bound is strictly positive when the subsystem is stable, and was shown in [13] to provide the least conservative estimate of tolerable perturbations when G_i is chosen to be the identity matrix in (4). When the subsystems are scalar, the following result is straightforward [13].

Lemma 1: For the stable scalar system

$$x_i[k+1] = a_{ii}x_i[k],$$

the robustness bound (5) with $G_i = 1$ is $\xi_v(1) = 1 - |a_{ii}|$. □

To analyze connective stability of the overall system, it is natural to consider a Lyapunov function that is constructed in terms of the Lyapunov functions for each of the individual subsystems as

$$v(x) = \sum_{i=1}^{N} c_i v_i(x_i), \quad (6)$$

where c_i is positive for each i. Using the above definition, the following result is derived in [13].

Theorem 1: The rate of decrease of the Lyapunov function given in (6), where each subsystem's Lyapunov function is of the form (3), can be upper-bounded as

$$v(x(k+1)) - v(x(k)) \leq -m^T W \bar{w}(x(k)) \ \ \forall x(k) \in \mathbb{R}^n$$

where

$$\begin{aligned} m &= \begin{bmatrix} c_1\lambda_{max}^{1/2}(H_1) & c_2\lambda_{max}^{1/2}(H_2) & \cdots & c_N\lambda_{max}^{1/2}(H_N) \end{bmatrix}^T \\ W_{ij} &= \begin{cases} \xi_v(G_i) & \text{if } i = j \\ -\|A_{ij}\|_2 & \text{if } i \neq j \end{cases} \\ \bar{w}(x(k)) &= \begin{bmatrix} \|x_1(k)\|_2 & \|x_2(k)\|_2 & \cdots & \|x_N(k)\|_2 \end{bmatrix}^T. \end{aligned} \quad (7)$$

□

In the matrix W defined by the above theorem, the system matrix A_{ii} of each subsystem i determines the robustness bound $\xi_v(G_i)$ from (4) and (5), and the worst-case strength of the coupling (corresponding to $e_{ij} = 1$) is factored in through $\|A_{ij}\|_2$. Based on the above characterization, the following result is immediately obtained from the first property of M-matrices described in Section II.

Theorem 2 ([8]): The system (1) is connectively stable if W is an M-matrix, where W is defined as in (7). □

The condition in the above theorem guarantees that there exists a set of positive scalars c_i such that the Lyapunov function (6) is decreasing for any value of $e_{ij} \in [0, 1]$ (and in fact, even for time-varying values of e_{ij} in this range). Thus, connective stability is a very strong property of coupled dynamical systems, and furthermore, the above test for connective stability is only sufficient in general. In order to gain more insight into topological properties that influence connective stability, and to quantify the conservativeness of the above test, we will focus on the case of *homogeneous scalar* subsystems in the rest of the paper; this will allow us to isolate the effect of the interconnection topology via tools from spectral graph theory.

IV. A Sufficient Condition for Connective Stability of Interconnected Scalar Systems

Suppose that the system $\mathcal{S}$ is composed of $N = n$ interconnected scalar subsystems of the form

$$x_i[k+1] = a_{ii}x_i[k] + \sum_{\substack{j=1 \\ j \neq i}}^{n} e_{ij}a_{ij}x_{ij}[k], \;\; 1 \leq i \leq n, \quad (8)$$

where $a_{ij} \in \mathbb{R}$, for all $i, j \in \{1, 2, \ldots, n\}$. If subsystem j is not connected to subsystem i, then $a_{ij} = 0$. As before, we will say that $\mathcal{S}$ is connectively stable if it is stable in the sense of Lyapunov for all $e_{ij} \in [0, 1]$. We will say that $\mathcal{S}$ is Schur stable if it is stable when $e_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, n\}$.

Assuming that $|a_{ii}| < 1$ for all i, using Lemma 1, the matrix W in (7) for the above coupled subsystems is given by

$$W_{ij} = \begin{cases} 1 - |a_{ii}| & \text{if } i = j \\ -|a_{ij}| & \text{otherwise.} \end{cases}$$

Let F be the $n \times n$ matrix with element (i, j) equal to a_{ij} from (8).[1] Thus, we have $W = I - |F|$. By Theorem 2, a sufficient condition for connective stability is that W is an M-matrix. By the second property of M-matrices in Section II, a necessary and sufficient condition for W to be an M-matrix is that all eigenvalues of $W = I - |F|$ have positive real parts. Since $|F|$ is a nonnegative matrix, the Perron-Frobenius theorem indicates $|F|$ has a positive real eigenvalue that has the largest magnitude of all the eigenvalues. Thus, all eigenvalues of W will have positive real parts if and only if all eigenvalues of $|F|$ have magnitude less than 1, leading to the following result.

Proposition 1: The system (8) with dynamics matrix F is connectively stable if $|F|$ is Schur stable. □

An immediate corollary of this result is as follows.

Corollary 1: In system (8), suppose $a_{ij} \geq 0$ for all $i, j \in \{1, 2, \ldots, n\}$. Then the system is connectively stable if and only if it is Schur stable. □

[1]We use F instead of A here in order to avoid confusion with the adjacency matrix that we will introduce later.

Thus, for nonnegative systems, Schur stability and the stronger notion of connective stability coincide regardless of the interconnection topology, and there is no conservativeness in the M-matrix characterization (see also [1], [14] for related discussions of this fact). Based on the above result, the only situation where conservatism can arise is when some of the elements a_{ij} are negative; this is perhaps not surprising, given that negative feedback loops are generally helpful in the stabilization of dynamical systems.

We consider here another case where connective stability coincides with Schur stability. First, suppose that the sign patterns in matrix F are symmetric (i.e., the sign of element (i, j) is the sign of element (j, i)), and that the graph is structurally balanced. In this case, matrix F is called a *Morishima matrix* [15], [16]. It can be shown that if F is a Morishima matrix, then the spectrum of F is the same as the spectrum of $|F|$; thus, for structurally balanced systems (which contain nonnegative systems as a special case), connective stability and Schur stability also coincide [14], [16].

In the rest of the paper, we will consider systems of the form (8) where $a_{ii} = a$ for all i, and $a_{ij} \in \{-b, 0, b\}$ for some $b \in \mathbb{R}$ and for all $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. In other words, we will assume that each subsystem has identical dynamics (given by the scalar a), and has identical coupling strength b to the other subsystems. If subsystem j is not connected to subsystem i, then $a_{ij} = 0$, and otherwise, $a_{ij} \in \{-b, b\}$ depending on whether the influence is positive or negative. We will also assume that $a_{ij} = a_{ji}$ (i.e., subsystems i and j influence each other in the same direction).[2] When $e_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, n\}$, the overall system S is given by

$$\mathcal{S}: \;\; x[k+1] = (aI + bA)\, x[k] \triangleq Fx[k], \quad (9)$$

where I is the $n \times n$ identity matrix and A is a signed adjacency matrix for the graph representing the overall topology of $\mathcal{S}$. We will take a and b to be nonnegative without loss of generality; the case where b is negative can be handled by simply absorbing the negative sign into A, and the case where a is negative can be handled using entirely symmetrical arguments to the ones that follow.

Proposition 1 yields the following sufficient condition for connective stability of (9).

Corollary 2: The system $\mathcal{S}$ in (9) is connectively stable if

$$a + b\lambda_{max}(|A|) < 1. \quad (10)$$

□

Since $|A|$ represents the unsigned adjacency matrix for the network, we can use tools from spectral graph theory to relate topological properties of the graph to condition (10). One such classical bound is as follows.

Proposition 2 ([17], [18]): Let $\mathcal{G}$ be a graph with maximum degree d_{max} and average degree d_{avg}. Then the largest

[2]While restrictive, these assumptions will allow us to gain insight into the role that the network topology plays in the stability of the system, something that is not easily obtained when considering arbitrary linear systems of the form (1).

eigenvalue of the unsigned adjacency matrix A satisfies

$$\max\{d_{avg}, \sqrt{d_{max}}\} \leq \lambda_{max}(A) \leq \max_{(v_i,v_j)\in\mathcal{E}} \sqrt{d_i d_j}. \quad (11)$$

□

Note that the upper bound in the above result is no larger than d_{max}, with equality if and only if two vertices of largest degree are neighbors in the graph. Using the above bounds in conjunction with Corollary 2 immediately yields the following graph-theoretic sufficient condition.

Corollary 3: Let the graph $\mathcal{G}$ associated with system $\mathcal{S}$ in (9) have maximum degree d_{max}. Then $\mathcal{S}$ is connectively stable if

$$a + b \max_{(v_i,v_j)\in\mathcal{E}} \sqrt{d_i d_j} < 1. \quad (12)$$

□

In the next section, we will provide bounds on the spectrum of F (based on the network structure and the sign pattern) that are necessary for Schur stability, and thus necessary for connective stability as well. This will allow us to quantify, using topological metrics, the conservativeness of condition (10) (and (12)) for connective stability, and generalize the structural balance results provided above.

V. Graph-Theoretic Conditions for Stability of Interconnected Scalar Systems

To study Schur stability of system (9), we will analyze the adjacency matrix A in its general signed form with $A \in \{-1, 0, +1\}^{n\times n}$. Denote the spectral radius of F by

$$\begin{aligned}\rho(F) &\triangleq \max\{\lambda_{max}(F), |\lambda_{min}(F)|\} \\ &= \max\{a + b\lambda_{max}(A), |a - b|\lambda_{min}(A)||\}. \quad (13)\end{aligned}$$

Schur stability of (9) requires $\rho(F) < 1$, and from the above expressions, this depends on the extreme eigenvalues of the signed adjacency matrix A. Thus, in order to relate these conditions to graph properties, we will start by providing some results on the spectrum of signed adjacency matrices, and then apply these results to infer stability properties of system (9).

A. Spectra of Signed Adjacency Matrices

The following result provides a graph-theoretic lower bound on the spectral radius of any signed adjacency matrix.

Proposition 3: For a signed symmetric adjacency matrix A we have

$$\rho(A) \geq \sqrt{d_{max}}, \quad (14)$$

where d_{max} is the maximum degree of the underlying graph.

□

Proof: First, note that $\rho(A^2) = \rho^2(A)$. Moreover note that the i-th diagonal element of A^2 is the square of the 2-norm of the i-th row of A; since each row consists of elements from $\{-1, 0, 1\}$, this is just the number of nonzero elements in the i-th row, which is the degree of node v_i. Suppose node v_j has the largest degree, and let e_j be a vector with all elements zero except for a single 1 in the j-th position. The Rayleigh quotient inequality [19] yields

$$\rho^2(A) = \rho(A^2) \geq \frac{e_j^T A^2 e_j}{e_j^T e_j} = e_j^T A^2 e_j = d_{max},$$

leading to the desired bound. ■

The observation that the diagonal elements of A^2 are the degrees of the nodes was also made in [20] and [21], although the subsequent derivations in the latter work led to a weaker lower bound of $\sqrt{d_{avg}}$ on $\rho(A)$. As pointed out in [21], the lower bound of $\sqrt{d_{avg}}$ is attained for a signing of a d_{avg}-regular graph. Since such graphs are also d_{max}-regular, this sharpness result carries forward to the above stronger bound.

While the above result provides a lower bound on the spectral radius for *any* signed matrix, we will be interested in how much the largest eigenvalue of a given signed adjacency matrix can deviate from the largest eigenvalue of the corresponding unsigned matrix. The next three results provide bounds on this deviation in terms of different topological properties.

First, recall from Section IV that if A is the signed adjacency matrix for a structurally balanced graph, then the spectrum of A is the same as the spectrum of $|A|$. Here, we provide a generalization of this result, and show that if a given graph is "almost balanced", the largest eigenvalue of the graph cannot differ too much from the largest eigenvalue of the unsigned graph. To do this, we will need the following concept.

Definition 3 ([22], [23]): The *frustration index* of a signed graph is the smallest number of edges that have to be removed to make the graph balanced. □

Proposition 4: Consider a signed graph $\mathcal{G}$ with frustration index k and let A be the associated signed adjacency matrix. The largest eigenvalue of A satisfies

$$\lambda_{max}(|A|) \geq \lambda_{max}(A) \geq \lambda_{max}(|A|) - 2\sqrt{2k}. \quad (15)$$

□

Proof: The upper bound follows from classical results pertaining to nonnegative matrices [19]. For the lower bound, if a signed graph $\mathcal{G}$ has frustration index k and signed adjacency matrix A, then there is a diagonal similarity transformation matrix S where each diagonal element is in $\{-1, 1\}$, such that the matrix $\bar{A} \triangleq SAS$ has exactly $2k$ negative elements [22]. Clearly the spectrum of A is the same as the spectrum of $\bar{A}$. We can now write $\bar{A} = |\bar{A}| - 2A_1$, where A_1 is the unsigned adjacency matrix for a graph with k edges corresponding to the negative elements in $\bar{A}$. Noting that $|\bar{A}| = |A|$ and applying Weyl's inequality [19] to the above equation provides

$$\begin{aligned}\lambda_{max}(A) = \lambda_{max}(\bar{A}) &\geq \lambda_{max}(|\bar{A}|) - 2\lambda_{max}(A_1) \\ &= \lambda_{max}(|A|) - 2\lambda_{max}(A_1).\end{aligned}$$

Using the fact that for an unsigned adjacency matrix $\mathcal{A}$ for a graph with m edges we have $\lambda_{max}(\mathcal{A}) \leq \sqrt{2m}$ [24], we obtain $\lambda_{max}(A_1) \leq \sqrt{2k}$ which yields the result. ■

When the graph is balanced (i.e., $k = 0$), the above result reduces to the fact that the largest eigenvalue of balanced graphs is equal to the largest eigenvalue of the unsigned graph. Since the largest eigenvalue of any unsigned graph is lower bounded by $\sqrt{d_{max}}$, signed graphs with large maximum degree and small frustration index will have largest eigenvalue relatively close to the corresponding balanced graph. In other words, small amounts of imbalance cannot dramatically change the largest eigenvalue.

Another bound for the largest (and smallest) eigenvalues of arbitrary signed graphs can be obtained from the cliques of the graph. Specifically, we will say a subgraph $\mathcal{H}$ of graph $\mathcal{G}$ is a positive (negative) clique if (1) $\mathcal{H}$ is a clique and (2) all of the signs on the edges are positive (negative).

Proposition 5: For a signed adjacency matrix A with associated signed graph $\mathcal{G}$, let the size of the largest positive clique be ω_+ and let the size of the largest negative clique be ω_-. We then have

$$\lambda_{max}(A) \geq \omega_+ - 1, \quad \lambda_{min}(A) \leq 1 - \omega_-. \tag{16}$$

□

Proof: By permuting columns and rows, one can recast A into the form

$$\begin{bmatrix} (J-I)_{\omega_+ \times \omega_+} & * \\ * & * \end{bmatrix},$$

where J is a square matrix consisting of all 1's, and $*$ represents the other elements of the adjacency matrix. Let

$$x = \frac{1}{\sqrt{\omega_+}} \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T,$$

where the first ω_+ elements are 1 and the rest are zero. Applying the Rayleigh quotient inequality, we have

$$\lambda_{max}(A) \geq \frac{x^T A x}{x^T x} = x^T A x = \frac{\omega_+(\omega_+ - 1)}{\omega_+} = \omega_+ - 1.$$

Using the same reasoning (i.e., by rearranging the adjacency matrix so that the largest negative clique appears in the first ω_- rows and columns), we obtain $\lambda_{min}(A) \leq 1 - \omega_-$. ■

The structural properties targeted by the above two results are NP-hard to characterize in general [25], [26]. The following result provides another lower bound for $\lambda_{max}(A)$ that depends only on the maximum number of negative edges incident on any node in the graph.

Proposition 6: Consider a signed graph $\mathcal{G}$ with associated signed adjacency matrix A. Let d_- be the largest number of negative elements in any row of A. Then

$$\lambda_{max}(|A|) \geq \lambda_{max}(A) \geq \lambda_{max}(|A|) - 2d_-. \tag{17}$$

□

Proof: Note that for any signed adjacency matrix A, one can write $A = |A| - 2B$, where B is the unsigned adjacency matrix with $B_{ij} = 1$ if and only if $A_{ij} = -1$. Once again applying Weyl's inequality provides

$$\begin{aligned} \lambda_{max}(A) &\geq \lambda_{max}(|A|) - 2\lambda_{max}(B) \\ &\geq \lambda_{max}(|A|) - 2d_{max}(B), \end{aligned}$$

where the last step follows from (11). The adjacency matrix B corresponds to the subgraph of $\mathcal{G}$ obtained by taking only the negative edges, and the largest degree in this subgraph is d_-, which yields the desired result. ■

The following three examples show there are graphs where each of the bounds in equations (15), (16) and (17) can outperform the other bounds. In each example, $\mathcal{G}$ is an undirected signed graph on n vertices that we will specify, and A is the corresponding signed adjacency matrix.

Example 1: Suppose $\mathcal{G}$ is a star centered at v_1. Let the signs of all edges be negative. The largest eigenvalue of $|A|$ for a star is $\sqrt{n-1}$ [27]. This is an acyclic graph and thus structurally balanced, with frustration index $k = 0$. The largest positive clique in this graph is of size 1, and the maximum degree of the subgraph containing the negative edges is $n-1$ (as it is the entire graph). Thus the frustration index bound (15) is exact, whereas the clique bound (16) predicts $\lambda_{max}(A) \geq 0$ and the maximum degree bound (17) predicts $\lambda_{max}(A) \geq \sqrt{n-1} - 2(n-1)$, both of which can be arbitrarily loose. □

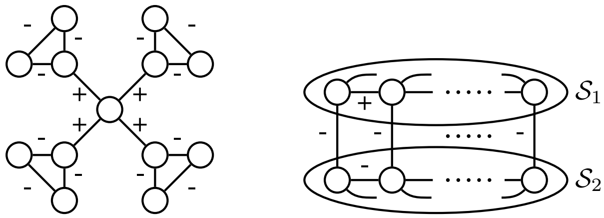
Example 2: Suppose $\mathcal{G}$ is the graph shown in Figure 1(a) consisting of $n = 3k + 1$ vertices for some $k \in \mathbb{N}$. There are k cycles of length 3, and each of the cycles has a single vertex connected to the central vertex v_1. All of the signs on edges in the cycles are negative, and the signs on the edges connecting v_1 to the cycles are positive. Since this graph has k negative cycles, one edge from each cycle must be removed in order to make the graph balanced, and thus the frustration index is k. The size of the largest positive clique is 2, and the maximum degree of the negative subgraph is 2. Thus the maximum degree bound (17) predicts that $\lambda_{max}(A) \geq \lambda_{max}(|A|) - 4$. The gap in the frustration index bound (15) is $2\sqrt{2k}$, which can be arbitrarily bad. Finally, the clique bound (16) predicts $\lambda_{max}(A) \geq 1$. By (11), we have $\lambda_{max}(|A|) \geq \sqrt{k}$. Thus the clique bound can also be arbitrarily bad. □

Example 3: Consider the graph $\mathcal{G}$ shown in Figure 1(b) with an even number n of vertices. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are cliques on $\frac{n}{2}$ vertices, and each vertex in set $\mathcal{S}_1$ has exactly one neighbor in $\mathcal{S}_2$ and vice versa. Suppose all of the signs on the edges in clique $\mathcal{S}_1$ are positive, and all of the signs in clique $\mathcal{S}_2$ are negative. The signs on the edges connecting $\mathcal{S}_1$ to $\mathcal{S}_2$ are negative. The maximum degree of the negative subgraph is $\frac{n}{2}$. To make this graph balanced, every odd cycle in $\mathcal{S}_2$ must be broken; there are at least $\lfloor \frac{n}{6} \rfloor$ disjoint triangles in $\mathcal{S}_2$, and thus the frustration index is at least $\lfloor \frac{n}{6} \rfloor$. The size of the largest positive clique is $\frac{n}{2}$. Since the graph is $\frac{n}{2}$-regular, we have $\lambda_{max}(|A|) = \frac{n}{2}$ [17]. The clique bound predicts $\lambda_{max}(A) \geq \frac{n}{2} - 1$, whereas the gaps in the other bounds increase with n. □

B. Applications to Schur Stability of Interconnected Systems

We can now apply our results on the spectrum of signed adjacency matrices to study the conservativeness of the condition for connective stability given in Corollary 2.

Proposition 7: Let $\mathcal{G}$ be the signed graph associated with the system $\mathcal{S}$ in (9), and let A be the corresponding signed adjacency matrix. Let k be the frustration index of the graph,



(a) (b)

Fig. 1: (a) Signed graph in Example 2 with $k = 4$. (b) Signed graph in Example 3. Sets $\mathcal{S}_1$ and $\mathcal{S}_2$ induce complete graphs on $\frac{n}{2}$ nodes. All edges inside $\mathcal{S}_1$ ($\mathcal{S}_2$) are positive (negative), and all edges between the sets are negative.

and let d_- be the largest number of negative edges incident to any node. Furthermore, let ω_+ denote the size of the largest positive clique in $\mathcal{G}$. Define

$$R = \min\{2\sqrt{2k}, 2d_-, \max_{(v_i,v_j)\in\mathcal{E}} \sqrt{d_i d_j} - \omega_+ + 1\}.$$

Then a necessary condition for Schur stability of $\mathcal{S}$ is

$$a + b(\lambda_{max}(|A|) - R) < 1.$$

□

Proof: The proof follows by noting that a necessary condition for stability of $\mathcal{S}$ is that $a + b\lambda_{max}(A) < 1$. We now apply the bounds in Propositions 4, 5 and 6. For the last bound, note from Proposition 5 that

$$\begin{aligned}\lambda_{max}(A) &\geq \omega_+ - 1 = \lambda_{max}(|A|) - (\lambda_{max}(|A|) - \omega_+ + 1)\\ &\geq \lambda_{max}(|A|) - (\max_{(v_i,v_j)\in\mathcal{E}} \sqrt{d_i d_j} - \omega_+ + 1),\end{aligned}$$

using (11). The result thus follows. ■

Comparing this to Corollary 2, we note the graph-theoretic parameter R in the above proposition quantifies the degree of conservativeness of the sufficient condition for connective stability. Loosely speaking, when the signed graph induced by the interconnections between the subsystems in (9) has a small frustration index, few negative edges incident to each node, or has a large positive clique compared to its max degree, the gap between the necessary condition and sufficient condition for connective stability also becomes small. Thus, for example, in the graph considered in Figure 1(b) the sufficient condition (10) for connective stability is

$$a + b\lambda_{max}(|A|) = a + b\frac{n}{2} < 1,$$

whereas the necessary condition for connective stability from Proposition 7 is $a + b\left(\frac{n}{2} - 1\right) < 1$. Since b must scale as $O(\frac{1}{n})$ in order for the necessary condition to be satisfied, we see that the sufficient and necessary conditions asymptotically coincide, despite the fact that this graph is far from structurally balanced.

VI. Conclusion

We studied a set of dynamical systems interconnected over a graph. For homogeneous scalar subsystems, we provided graph-theoretic bounds on the dynamics and coupling strengths in order for the system to be stable, both in a connective sense and in the classical Schur sense. To capture the effect of the signs of interconnections, we developed bounds on the largest eigenvalue of signed adjacency matrices, and provided conditions under which the sufficient conditions and necessary conditions for connective stability coincide. An important and challenging area for future research is to extend these results to more general heterogeneous systems.

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