Controlling Human Utilization of Shared Resources via Taxes

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***Abstract*— We study a general resource sharing game where overutilization by selfish decision makers leads to possible failure of the resource. Our goal is to understand the effectiveness of a taxation mechanism in reducing the utilization and fragility of the resource when players have behavioral risk preferences. In particular, we incorporate risk preferences drawn from *prospect theory*, an empirically validated behavioral model of human decision making. We first identify counter-intuitive behavior under prospect theory, where utilization (and hence fragility) can *increase* under taxation, depending on the resource characteristics. We then identify conditions under which taxation is effective in reducing the fragility of the resource. We also show that homogeneous sensitivities to taxes leads to smaller failure probability compared to the case where players have heterogeneous (player-specific) sensitivities to taxes.**

I. Introduction

There are many settings where self-interested individuals compete over a shared resource, both natural (such as fisheries and groundwater [1]), and engineered (such as transportation networks [2]). Game-theoretic analyses of resource sharing have been carried out in multiple disciplines, including economics [1], psychology [3], engineering [4] and computer science [5]. Controlling the resource utilization levels through economic incentives such as taxes and rewards has been studied extensively [6]–[8].

In many of the above resource sharing settings, the utilities of the players are uncertain, possibly due to resource failure caused by overutilization [3], [9], [10]. When making decisions under uncertainty, the *risk preferences* of the individuals often have a significant impact on their behavior. Most of the existing work on uncertainty in resource sharing settings (with the exception of our recent work [11]) models decision makers as risk neutral (expected value maximizers) or risk averse (expected utility maximizers with respect to a concave utility function).

Empirical evidence, however, has shown that the preferences of human decision makers systematically deviate from the preferences of a risk neutral or risk averse decision maker [12]. For example, humans compare different outcomes with a reference utility level, and exhibit different attitudes towards *gains* and *losses*. In their Nobel-prize winning work, Kahneman and Tversky proposed *prospect theory* [13] in order to capture these attitudes with appropriately defined utility and probability weighting functions.[1]

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[1]The probability weighting function captures the transformation of true probabilities into perceived probabilities by human decision makers. In this work, we focus exclusively on the transformation of the utility function.

In this paper, we investigate how behavioral risk attitudes affect users' decisions in resource sharing settings. Specifically, we focus on how prospect-theoretic decision makers respond to tax mechanisms imposed by central planners. We build upon our prior work in [11], where we analyzed the impact of prospect-theoretic preferences on users' equilibrium strategies in the absence of taxation.

We consider the following game-theoretic setting. A set of players independently choose their level of utilization of (or "investment" in) a shared resource. As the total investment by all players increases, it becomes more likely for the resource to fail, causing the players to lose their investments. If the resource does not fail, then the players receive a return per unit investment according to a rate of return function. We refer to this setting as a *resource sharing game*.

Resources with increasing rates of return exhibit so-called *network effects* [14]; examples include shared online platforms such as peer to peer file sharing systems, where increased use leads to greater return. However, there are instances where authorities have shut down large online platforms, such as illegal peer to peer file sharing systems [15] and websites encouraging terrorism [16]. This is captured by resource failure in our setting. Resources with decreasing rates of return model *congestion effects* and describe engineered systems such as communication networks [4] and open-access natural resources such as fisheries [1]. We consider both network and congestion effects in our analysis.

To control utilization, we study a tax mechanism where a player is charged a tax amount proportional to her investment in the resource. It is perhaps natural to expect that a higher tax rate will reduce the utilization and hence failure probability (or *fragility*) of the resource at a pure Nash equilibrium (PNE). However, perhaps counter-intuitively, we find that behavioral risk preferences can sometimes cause utilization (and fragility) to *increase* with a higher tax rate. Therefore, the goal of this paper is provide an understanding of how prospect-theoretic risk attitudes influence utilization and fragility of the resource under taxation.

We first demonstrate through examples that risk seeking behavior in losses (a characteristic of prospect theory) can cause utilization (and fragility) to potentially increase with a higher tax rate. Then, we identify values of certain parameters in the prospect-theoretic utility function for which taxation is effective in reducing fragility. Finally, we consider player-specific sensitivities to the tax rate. We prove that when players are loss averse (formally defined in Section II), homogeneous sensitivities to the tax rate leads to a smaller fragility compared to the heterogeneous case.

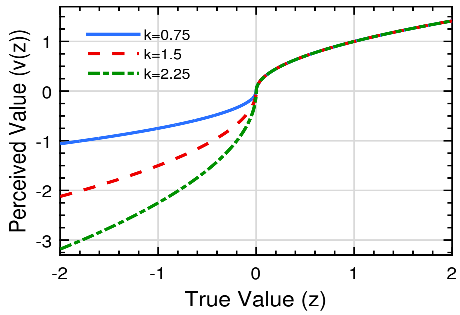


Fig. 1: Prospect-theoretic utility function (1) with $\alpha = 0.5$ and reference point $z_0 = 0$.

II. Prospect Theory

As discussed in the previous section, our focus will be on behavioral preferences captured by *prospect theory* [13]. Specifically, consider a gamble that has an outcome with value $z \in \mathbb{R}$. A prospect-theoretic individual *perceives* this value in a skewed manner, via the function

$$v(z) = \begin{cases} (z - z_0)^\alpha, & \text{when } z \geq z_0 \\ -k(z_0 - z)^\alpha & \text{otherwise}, \end{cases} \quad (1)$$

where z_0 is the reference point, $\alpha \in (0, 1]$ is the *sensitivity parameter* and $k \in (0, \infty]$ is referred to as the *loss aversion index*. Increase in utility with respect to the reference point ($z \geq z_0$) is referred to as a *gain* and decrease in utility is referred to as a *loss* ($z < z_0$).

The parameter α shapes the utility function according to observed behavior, i.e., the utility function is concave in the domain of gains and convex in the domain of losses. Accordingly, the decision maker is said to be "risk averse" in gains and "risk seeking" in losses. As its name indicates, the parameter k captures loss aversion behavior. Specifically, when $\alpha = 1$, a loss of \$1 *feels* like a loss of \$$k$ to the player. A value of $k > 1$ implies that the individual is *loss averse*, while $k < 1$ implies that the individual is *gain seeking*. When the reference point is an exogenous constant, the values $k = 1$ and $\alpha = 1$ capture risk neutral behavior. A smaller α implies greater deviation from risk neutral behavior. The shape of the value function is shown for different values of k in Figure 1, with $\alpha = 0.5$ and $z_0 = 0$.

III. Resource Sharing Game

Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of players. Each player decides how much to invest in a shared (common pool) resource (CPR). The investment of player i in the CPR is denoted as x_i, where $x_i \in [0, 1]$. The total investment by all players in the CPR is denoted as $x_T = \sum_{i \in \mathcal{N}} x_i$.

The return from the CPR is subject to risk, captured by a probability of failure $p(x_T)$, which is a function of the aggregate investment in the resource. If the CPR fails, the players lose their investments in the CPR. If the CPR does not fail, it has a rate of return as a function of the total investment x_T, denoted as $r(x_T)$; in other words, player i gets $x_i r(x_T)$ from the CPR when it does not fail and $-x_i$ if it fails. Each player is prospect-theoretic, with a player-specific loss aversion index $k_i \in \mathbb{R}_{>0}$ and sensitivity parameter $\alpha_i \in (0, 1]$. We assume that the reference utility is equal to 0 and is derived when the player invests 0 in the CPR.[2]

Let $y_i = \sum_{j=1, j \neq i}^{n} x_j$, $x_j \in [0, 1]$, be the total investment of all players other than i. Using the prospect-theoretic utility function (1), player i's *perception* of the gains and losses from the CPR is given by

$$u_i(x_i, y_i) = \begin{cases} (x_i r(x_i + y_i))^{\alpha_i}, & \text{with prob. } 1 - p(x_i + y_i) \\ -k_i x_i^{\alpha_i}, & \text{with prob. } p(x_i + y_i). \end{cases} \quad (2)$$

The players are expected utility maximizers with respect to the utility function given by equation (2) and thus maximize

$$\begin{aligned} \mathbb{E}(u_i) &= (x_i r(x_T))^{\alpha_i}(1 - p(x_T)) - k_i x_i^{\alpha_i} p(x_T) \\ &= x_i^{\alpha_i}[r(x_T)^{\alpha_i}(1 - p(x_T)) - k_i p(x_T)] \\ &\triangleq x^{\alpha_i} f_i(x_T). \end{aligned} \quad (3)$$

Here $f_i(x_T)$ is referred to as the *effective rate of return* of player i. We denote this resource sharing game as $\Gamma(\mathcal{N}, \{u_i\}_{i \in \mathcal{N}})$. We make the following assumptions.

Assumption 1: The class of resource sharing games $\Gamma(\mathcal{N}, \{u_i\}_{i \in \mathcal{N}})$ has the following properties.

1) The failure probability $p(x_T)$ is convex, strictly increasing and continuously differentiable for $x_T \in [0, 1)$ and $p(x_T) = 1$ for $x_T \geq 1$.
2) The rate of return $r(x_T)$ is positive, concave, strictly monotonic and continuously differentiable.

These assumptions capture a fairly broad class of resources, while making the analysis tractable.

IV. Taxation

In order to control the amount of utilization (and fragility) of the shared resource (CPR), we investigate the impact of a uniform tax rate t per unit investment in the CPR, i.e., a player i with investment x_i in the CPR is charged $t x_i$ as tax. As before, the reference utility is 0, which arises when $x_i = 0$. When the CPR does not fail, the reference dependent utility of player i is $(x_i(r(x_T) - t))^{\alpha_i}$. When the CPR fails, the magnitude of loss is $(1 + t)x_i$, which is the sum of the wealth x_i invested in the CPR, and the tax $t x_i$. The modified reference dependent expected utility function is

$$\begin{aligned} \mathbb{E}(u_i) &= x_i^{\alpha_i}(r(x_T) - t)^{\alpha_i}(1 - p(x_T)) \\ &\quad - k_i x_i^{\alpha_i}(1 + t)^{\alpha_i} p(x_T) \\ &= x_i^{\alpha_i}[(r(x_T) - t)^{\alpha_i}(1 - p(x_T)) - k_i(1 + t)^{\alpha_i} p(x_T)] \\ &\triangleq x_i^{\alpha_i} f_i(x_T, t). \end{aligned} \quad (4)$$

The effective rate of return $f_i(x_T, t)$ is now a function of the tax rate t in addition to x_T.

The above utility function is analogous to the original utility function in equation (3) with a modified rate of return function, $\tilde{r}(x_T) \triangleq r(x_T) - t$, and a modified loss aversion

[2] The above formulation is equivalent to an instance of *atomic splittable congestion games* [4], [5] over a network with two nodes and two parallel links (resources) joining them. One link corresponds to the CPR described above and the second one has a constant delay of 1. The reference utility is the utility of a player when she sends her entire traffic by the second link.

index $k_i(1+t)^{\alpha_i}$. We denote $\bar{t} \triangleq \min_{x_T\in[0,1]} r(x_T)$, i.e., it is the infimum tax rate at which $\tilde{r}(x_T) < 0$ for some $x_T \in [0,1]$. In this paper, we restrict ourselves to tax rates in $[0,\bar{t})$ for convenience. For a given $t \in [0,\bar{t})$, we have the following result on the existence and uniqueness of a PNE.

Theorem 1: A resource sharing game satisfying Assumption 1 and any fixed tax rate $t \in [0,\bar{t})$ possesses a unique pure Nash equilibrium (PNE).

The proof follows the same arguments as for the case without taxation in [11]. In particular, for any fixed tax rate $t \in [0,\bar{t})$, if $r(x_T)$ has the properties defined in Assumption 1 (i.e., monotone, positive and concave), then $\tilde{r}(x_T)$ retains those properties. In other words, for any fixed t, the function $f_i(x_T,t)$ in (4) has the same properties (of interest) as the function $f_i(x_T)$ in (3). As a result, our results on the uniqueness and continuity of best responses from [11] for the utility function (3) carry over to the modified game with taxation. The existence of a PNE in the resource sharing game at a fixed tax rate follows from Brouwer's fixed point theorem [17]. The uniqueness follows from the monotonicity of nonzero best responses in the total investment from [11].

Having established the existence of PNE under any fixed tax rate, we now study how the characteristics of the PNE are affected by the tax rate.

V. Higher Tax Rate Can Increase Utilization

We start our investigation with the following example that considers resources that exhibit network effects (captured by increasing $r(x_T)$). As we discussed earlier, online platforms such as peer to peer file sharing systems exhibit such effects.

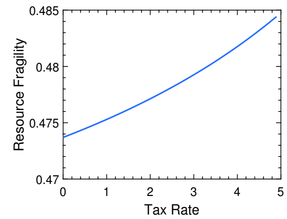
Example 1: Consider a resource sharing game with $n = 3$ players. Let the rate of return of the resource be $r(x_T) = 10x_T + 5$ and the failure probability of the resource be $p(x_T) = x_T^4$. Let $\alpha = 1$ and $k = 0.05$ for all the players, i.e., the players are gain seeking.

Figure 2a shows a plot of the failure probability $p(x_T)$ of the resource at the PNE investment x_T as the tax rate is swept from $t = 0$ to $t = 4.9$. As shown in the figure, the utilization and failure probability of the CPR at the corresponding PNE increases with tax rate. □

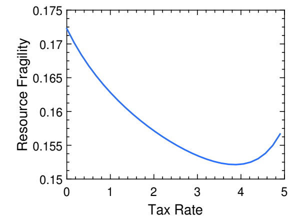
In the above example, imposing a higher tax rate to discourage utilization instead leads to larger investment and fragility of the resource. When the tax rate is increased, the modified return $\tilde{r}(x_T) = r(x_T) - t$ decreases. Since $r(x_T)$ is increasing in x_T, players increase their investment to compensate for the decrease in $\tilde{r}(x_T)$ due to higher tax rate. While this increase in investment also causes the failure probability to increase, the magnitude of $p(x_T) = x_T^4$ remains relatively small when x_T is bounded away from 1. Furthermore, since players have a small loss aversion index $k = 0.05$, the (dis)utility from a loss is small compared to the increase in utility due to the growing $r(x_T)$.

The following example shows that such behavior is not limited to the case of players with $k < 1$.

Example 2: Consider a resource sharing game with $n = 3$ players. Let the rate of return of the resource be $r(x_T) = 8x_T + 5$ and the failure probability of the resource be



(a) $k = 0.05$ and $\alpha = 1$



(b) $k = 1.2$ and $\alpha = 0.15$

Fig. 2: Fragility increases with tax rate due to prospect-theoretic preferences (with parameters shown above).

$p(x_T) = x_T$. Let $\alpha = 0.15$ and $k = 1.2$ for all the players, i.e., all the players are loss averse, and the deviation from risk neutral behavior ($\alpha = 1$ and $k = 1$) is significant. As shown in Figure 2b, when the tax rate increases from $t = 0$ to $t = 4.9$, the fragility is not monotonically decreasing. □

Recall from Figure 1 that values of $\alpha < 1$ gives rise to risk seeking behavior in losses and risk averse behavior in gains. When the value of α approaches 0, the modified loss aversion index $k(1+t)^{\alpha}$ does not increase by much at a higher tax rate. Similar to the discussion after Example 1, this encourages the players to increase their investment into the CPR with network effects.

In the subsequent sections, we will provide conditions under which the fragility (and utilization) will decrease at a higher tax rate. The proofs of our analytical results heavily rely on certain properties of the best response. We thus start by introducing these preliminary results.

VI. Properties of the Best Response

Consider a resource sharing game with tax rate $t \in [0,\bar{t})$. For a player i, let y_i be the total investment in the CPR by all players other than i. The *best response* of player i is the investment that maximizes her utility for a given y_i, and is denoted as $b_i(y_i)$. The utility in (4) is not concave (in general) over the entire strategy space of a player. Nonetheless, the following lemma shows that for a given y_i and fixed t, $b_i(y_i)$ is unique. The proof follows from the analysis of the best responses in non taxed settings from [11].

Lemma 1: [11] For a player i in a resource sharing game satisfying Assumption 1 and with a fixed tax rate t, the following are true.

1) There exists a $\bar{y}_i \in [0,1]$, depending on the tax rate t, such that 0 is a best response if and only if $y_i \geq \bar{y}_i$.
2) When $\bar{y}_i > 0$, $f_i(\bar{y}_i, t) = 0$ and there exists an interval $\mathcal{I}_i^t \subset [0,\bar{y}_i)$ such that if $y_i < \bar{y}_i$, then there exists a unique positive best response $b_i(y_i)$ that satisfies $b_i(y_i) + y_i \in \mathcal{I}_i^t$.
3) For $x_T \in \mathcal{I}_i^t$, we have $f_i(x_T,t) > 0$ and $f_{i,x}(x_T,t) \triangleq \frac{\partial f_i(x_T,t)}{\partial x_T} < 0$.

From Assumption 1, we observe that irrespective of the value of y_i, it is never an optimal strategy for a player to invest $x_i = 1$ in the resource, since it leads to certain failure of the resource (and a loss for the player). Therefore, from (4), when the best response of a player is nonzero, it must satisfy the first order condition $\frac{\partial \mathbb{E}(u_i)}{\partial x_i} = 0$, leading to

$$x_i f_{i,x}(x_T, t) + \alpha_i f_i(x_T, t) = 0 \tag{5}$$

$$\Longrightarrow x_i = \frac{\alpha_i f_i(x_T, t)}{-f_{i,x}(x_T, t)} \triangleq g_i(x_T, t), \tag{6}$$

where $x_T = x_i + y_i$. We have the following result on the monotonicity of the function $g_i(x_T, t)$ with respect to x_T.

Lemma 2: [11] If $x_T \in \mathcal{I}_i^t$ defined in Lemma 1, then $\frac{\partial g_i(x_T,t)}{\partial x_T} < 0$.

As we will show later, $g_i(x_T, t)$ is not always decreasing in t. We present a helpful characterization of the total investment x_T^* at the PNE as follows.

Definition 1: We define the support of the PNE of a resource sharing game Γ as the set of players with nonzero investment at the PNE, and denote it as $Supp(\Gamma)$. □

With the above definition and equation (6), we have

$$\sum_{i \in Supp(\Gamma)} g_i(x_T^*, t) = x_T^*. \tag{7}$$

We also require the following lemma for the subsequent analysis. Let Γ_1 and Γ_2 be two instances of a resource sharing game, with tax rates t_1 and t_2, respectively. Let $r(\cdot)$ and $p(\cdot)$ satisfy Assumption 1. Denote the total investments at the PNEs of the two games by x_{T_1} and x_{T_2}, respectively.

Lemma 3: If $t_1 > t_2$ and $x_{T_1} > x_{T_2}$, we have $Supp(\Gamma_1) \subseteq Supp(\Gamma_2)$.

Proof: When $r(x_T)$ is decreasing, the effective rate of return (4) for player i, $f_i(x_T, t)$, is decreasing in both of its arguments. Thus, whenever $t_1 > t_2$ and $x_{T_1} > x_{T_2}$, $f_i(x_{T_2}, t_2) \geq f_i(x_{T_1}, t_1)$. If player $j \in Supp(\Gamma_1)$, then $f_j(x_{T_1}, t_1) > 0$. It follows that $f_j(x_{T_2}, t_2) > 0$ and $j \in Supp(\Gamma_2)$.

When $r(x_T)$ is increasing, direct calculation shows that $f_i(x_T, t)$ is concave in x_T whenever $\tilde{r}(x_T) > 0$. If player $i \notin Supp(\Gamma_2)$, then it follows from Lemma 1 that $x_{T_2} \in [\bar{y}_i, 1]$. From the definition of $\bar{y}_i$, we have $f_i(x_{T_2}, t_2) < 0$. Furthermore, since f is concave, we have $f_{i,x}(x, t_2) < 0, \forall x > \bar{y}_i$. Therefore $f_i(x_{T_1}, t_2) < 0$ since $\bar{y}_i < x_{T_2} < x_{T_1}$. As a result, $f_i(x_{T_1}, t_1) < 0$, and $i \notin Supp(\Gamma_1)$. ■

We are now ready to analyze the impact of the tax rate on resource utilization under increasing $r(x_T)$ (Section VII) and decreasing $r(x_T)$ (Section VIII).

VII. Increasing Rate of Return

We first consider shared resources with increasing $r(x_T)$. We focus on understanding the impact of loss aversion on resource utilization levels under taxation, and therefore consider players with $\alpha = 1$ in this section. We obtain a sufficient condition in terms of the smallest loss aversion index among all players such that a higher tax rate leads to smaller total investment.

Without loss of generality, let players be ordered such that $k_1 \leq k_2 \leq \ldots \leq k_n$. When $\alpha_i = 1$ for all players i, the effective rate of return (from (4)) and its derivative with respect to the total investment are

$$f_i(x, t) = \hat{f}_i(x) - t(1 - p(x) + k_i p(x)), \tag{8}$$

$$f_{i,x}(x, t) = \hat{f}_i'(x) - t(k_i - 1)p'(x), \tag{9}$$

with $\hat{f}_i(x) \triangleq r(x)(1 - p(x)) - k_i p(x)$ being the effective rate of return function with $\alpha_i = 1$ and tax rate $t = 0$.

Note that when $r(x_T)$ is increasing, the interval $\mathcal{I}_i^t$ defined in Lemma 1 is $[\hat{z}_i, \bar{y}_i]$ where $\hat{z}_i := \inf\{z \in (0,1) | f_{i,x}(z, t) < 0\}$, i.e., $f_{i,x}(\hat{z}_i, t) = 0$ and $f_i(\bar{y}_i, t) = 0$ [11].

Proposition 1: Consider a resource sharing game with increasing $r(x_T)$ and let all players have $\alpha = 1$. Let the total investment at the PNE at a tax rate t_2 be x_{T_2}. Suppose

$$1 - k_1 < \frac{-\hat{f}_1'(x_{T_2})}{(\hat{f}_1(x_{T_2})p'(x_{T_2}) - \hat{f}_1'(x_{T_2})p(x_{T_2}))}.$$

Then, at any tax rate t_1 with $t_1 > t_2$, the corresponding total PNE investment x_{T_1} satisfies $x_{T_1} \leq x_{T_2}$. In particular, for $k_1 \geq 1$, utilization always decreases with tax rate.

Proof: Assume on the contrary that $x_{T_1} > x_{T_2}$ for some tax rate $t_1 > t_2$. According to Lemma 3, we have $Supp(\Gamma_1) \subseteq Supp(\Gamma_2)$. From the characterization of PNE in equation (7), we obtain

$$\sum_{j \in Supp(\Gamma_1)} g_j(x_{T_1}, t_1) > x_{T_2} = \sum_{j \in Supp(\Gamma_2)} g_j(x_{T_2}, t_2)$$

$$\Longrightarrow \sum_{j \in Supp(\Gamma_1)} g_j(x_{T_1}, t_1) > \sum_{j \in Supp(\Gamma_1)} g_j(x_{T_2}, t_2). \tag{10}$$

In the remainder of the proof, our goal is to contradict the inequality in equation (10). In particular, for a player $j \in Supp(\Gamma_1)$, we show that $g_j(x_{T_1}, t_1) < g_j(x_{T_2}, t_2)$.

Consider a player $j \in Supp(\Gamma_1)$. Let $\hat{z}_1$ and $\hat{z}_2$ be such that $f_{j,x}(\hat{z}_1, t_1) = 0$ and $f_{j,x}(\hat{z}_2, t_2) = 0$. Similarly, let $\bar{y}_1$ and $\bar{y}_2$ be such that $f_j(\bar{y}_1, t_1) = 0$ and $f_j(\bar{y}_2, t_2) = 0$. From equation (8), we know that $f_j(x, t)$ is strictly decreasing in t, which implies $f_j(\bar{y}_1, t_2) > 0$. Thus, $\bar{y}_1 < \bar{y}_2$. We thus have $\hat{z}_2 < x_{T_2} < x_{T_1} < \bar{y}_1 < \bar{y}_2$ and $[x_{T_2}, x_{T_1}] \subset \mathcal{I}_j^{t_2}$. From our result on the monotonicity of $g_j(x_T, t)$ in x_T in Lemma 2, we obtain $g_j(x_{T_1}, t_2) < g_j(x_{T_2}, t_2)$.

It remains to show that $g_j(x_{T_1}, t_1) < g_j(x_{T_1}, t_2)$. If $k_j \geq 1$, it is easy to see from equations (8) and (9) that $f_j(x, t)$ is decreasing in t and $-f_{j,x}(x, t)$ is non decreasing in t.

It is now sufficient to show that when $k_j < 1$, $\frac{\partial g_j(x_{T_1},t)}{\partial t} < 0$ for $t \in [t_2, t_1]$ when k_j satisfies the condition stated in the proposition. We drop the subscript j in the following computations. Following (8) and (9), we compute the partial derivatives of $f(x, t)$ and $f_x(x, t)$ with respect to t as

$$f_t(x, t) = -(1 - p(x)) - kp(x), f_{x,t}(x, t) = (1 - k)p'(x).$$

Note that $\frac{\partial g(x,t)}{\partial t} = \frac{f(x,t) f_{x,t}(x,t) - f_t(x,t) f_x(x,t)}{(f_x(x,t))^2}$. We compute the numerator while dropping the arguments x and t for ease of notation. In particular,

$$\begin{aligned} &f f_{x,t} - f_t f_x \\ &= \hat{f}(1 - k)p' - t(1 - p + kp)(1 - k)p' + \hat{f}'(1 - p + kp) \\ &\qquad - t(k - 1)p'(1 - p + kp) \\ &= \hat{f}' - (1 - k)\hat{f}'p + (1 - k)\hat{f}p'. \end{aligned}$$

The requirement $\frac{\partial g(x_{T_1},t)}{\partial t} < 0$ is equivalent to

$$1 - k < \frac{-\hat{f}'}{\hat{f}p' - \hat{f}'p} = \frac{-\hat{f}'}{rp' - r'p(1 - p)} \triangleq h(x_{T_1}). \tag{11}$$

The above inequality holds because $\hat{f}_j(x_{T_1}) > 0$ and $\hat{f}'_j(x_{T_1}) < 0$.[3] A direct computation shows that $h(x)$ is an increasing function of x when $f(x) > 0$ and $r(x)$ is concave. Now, suppose we have $1 - k_j < h_j(x_{T_2})$ for every player $j \in Supp(\Gamma_1)$ with $k_j < 1$. Since $x_{T_1} > x_{T_2}$ according to our contradiction hypothesis, we obtain $1 - k_j < h_j(x_{T_1})$. Accordingly, $\frac{\partial g(x_{T_1},t)}{\partial t} < 0$ for $t \in [t_2, t_1]$, and we have $g_j(x_{T_1}, t_1) < g_j(x_{T_1}, t_2)$, which is the desired contradiction.

In resource sharing games with $\alpha = 1$, the smallest loss aversion index k_1 is part of the support at every PNE with nonzero total investment [11]. The L.H.S. of equation (11) is decreasing in k, while the R.H.S. is increasing in k. Therefore, if equation (11) is satisfied for k_1, every other $k_1 \leq k < 1$ satisfies it. This concludes the proof. ■

The above result states that when the users are risk neutral or loss averse (i.e., have $k_i \geq 1$) and have $\alpha_i = 1$, imposing a higher tax rate leads to a decrease in the total investment (and therefore fragility) at the PNE. On the other hand, if certain players are gain seeking (with $k_i < 1$), then at a given tax rate t with a resulting total investment in the CPR x_T^*, we can check the stated condition to determine if a higher tax rate will lower the total investment.

VIII. Decreasing Rate of Return

In Section V, we showed that in games with increasing $r(x_T)$, behavioral attitudes can impact users' investment strategies in the presence of taxation in counter-intuitive ways. In contrast, for resources with a decreasing $r(x_T)$, we show here that an increase of tax rate always leads to a reduced total PNE investment. The result holds under heterogeneous risk preferences among players, i.e., when $k_i \in \mathbb{R}_{>0}$ and $\alpha_i \in (0, 1]$ are player-specific.

We start with the following useful lemma, which holds for the general form of the utility function (4) with $\alpha_i \in (0, 1]$. The proof is mostly an exercise in algebra and is omitted.

Lemma 4: Consider $g_i(x,t) := \frac{\alpha_i f_i(x,t)}{(-f_{i,x}(x,t))}$ as defined in (6). If $f_i(x,t) > 0$, we have $\frac{\partial g_i(x,t)}{\partial t} < 0$.

Proposition 2: Consider a resource sharing game with decreasing $r(x_T)$ where all players have $\alpha \in (0, 1]$ and $k > 0$. Then, the total investment at the PNE is decreasing in the tax rate, i.e., if $t_1 > t_2$, we have $x_{T_1} \leq x_{T_2}$.

Proof: Assume on the contrary that $x_{T_1} > x_{T_2}$. From Lemma 3 and following identical arguments as the proof of Proposition 1, we can show that (10) holds in our case, i.e.,

$$\sum_{j \in Supp(\Gamma_1)} g_j(x_{T_1}, t_1) > \sum_{j \in Supp(\Gamma_1)} g_j(x_{T_2}, t_2). \quad (12)$$

We obtain a contradiction to the above equation as follows.

Consider a player $j \in Supp(\Gamma_1)$. Then $x_{T_1} \in \mathcal{I}_j^{t_1}$, where $\mathcal{I}_j^{t_1}$ is the interval such that for $x \in \mathcal{I}_j^{t_1}$, we have $f_j(x, t_1) > 0$ and $f_{j,x}(x, t_1) < 0$, following Lemma 1. When $r(x)$ (and accordingly $f_j(x,t)$) is decreasing in x and $x_{T_1} > x_{T_2}$, then it is easy to see that $x_{T_2} \in \mathcal{I}_j^{t_1}$. Therefore, from Lemma 2 we have $g_j(x_{T_1}, t_1) < g_j(x_{T_2}, t_1)$.

Now for player $j \in Supp(\Gamma_1)$, $f_j(x_{T_1}, t_1) > 0$. Under our assumption that $x_{T_2} < x_{T_1}$, we obtain $f_j(x_{T_2}, t_1) > 0$ since $f_j(x,t)$ is decreasing in x. Since $f_j(x,t)$ is also decreasing in the second argument t, $f_j(x_{T_2}, t) > 0$ for $t \in [t_2, t_1]$. As a result, from Lemma 4, we have $g_j(x_{T_2}, t_1) < g_j(x_{T_2}, t_2)$. Thus, for every player $j \in Supp(\Gamma_1)$, $g_j(x_{T_1}, t_1) < g_j(x_{T_2}, t_2)$, which contradicts equation (12). ■

[3] From equation (8), we note that $f_j(x,t)$ is strictly decreasing in t. Since $f_j(x_{T_1}, t_1) > 0$, we must have $f_j(x_{T_1}, 0) > 0$, i.e., $\hat{f}_j(x_{T_1}) > 0$. Similarly, since x_{T_1} is the total PNE investment at the tax rate t_1, we have $0 > f_{j,x}(x_{T_1}, t_1) = \hat{f}'_j(x_{T_1}) - t_1(k_j - 1)p'(x_{T_1}) > \hat{f}'_j(x_{T_1})$.

IX. Heterogeneity in Tax Sensitivities

In this section, we examine the effect of heterogeneous sensitivities to the tax rate on the fragility of the resource. The sensitivity of player i is captured by a parameter $\gamma_i \in [0, 1]$. Accordingly, for a tax rate t, player i experiences (or perceives) an equivalent tax rate of $\gamma_i t$. Note that the impact of tax sensitivities on price of anarchy was studied recently in [7], outside of the context of behavioral risk attitudes.

In order to isolate the effect of heterogeneous tax sensitivities, we assume that all players have identical loss aversion indices k and $\alpha = 1$. Following equation (4), the expected utility of player i is defined as

$$\begin{aligned}\mathbb{E}(u_i) &= x_i(r(x_T) - \gamma_i t)(1 - p(x_T)) - k(1 + \gamma_i t)x_i p(x_T) \\ &= x_i[\hat{f}(x_T) - t\gamma_i(1 + (k-1)p(x_T))] \\ &\triangleq x_i[\hat{f}(x_T) - \gamma_i t v(x_T)] \triangleq x_i f(x_T, \gamma_i t), \quad (13)\end{aligned}$$

where $\hat{f}(x_T) \triangleq r(x_T)(1 - p(x_T)) - kp(x_T)$, and $v(x_T) \triangleq 1 + (k-1)p(x_T)$.

In the following result, we show that when the players are loss averse, i.e., $k > 1$, the total investment at the PNE is smaller when the players have homogeneous tax sensitivities, compared to the PNE of a game where the tax sensitivities are heterogeneous. The result holds for both increasing and decreasing rate of return functions.

Proposition 3: Let $\mathbf{\Gamma}_m$ be the set of resource sharing games with n players and the following characteristics. Let the resource be the same across all games in $\mathbf{\Gamma}_m$, with $r(x_T)$ and $p(x_T)$ satisfying Assumption 1. Let $\alpha = 1$ and $k > 1$ for all players and let the mean of the sensitivities in each game be γ_m. Then, for any tax rate $t \in [0, \bar{t})$, among all games in $\mathbf{\Gamma}_m$, the instance where all players have identical tax sensitivity γ_m has the smallest total PNE investment.

Proof: Let $\Gamma_H, \Gamma_M \in \mathbf{\Gamma}_m$ be two instances of resource sharing games, with respective total PNE investments being x_T^H and x_T^M. In Γ_M, all players have sensitivity parameters equal to γ_m, while in Γ_H, the sensitivity parameters $(\gamma_i, i \in \{1, 2, \ldots, n\})$ are player-specific. Without loss of generality, let $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq 1$, with $\sum_{i=1}^n \gamma_i = n\gamma_m$.

Suppose $x_T^H = 0$. Then, we have $\hat{f}(x_T) - \gamma_1 t v(x_T) \leq 0$ for $x_T \in [0, 1]$. Since $\gamma_m \geq \gamma_1$, we also have $\hat{f}(x_T) - \gamma_m t v(x_T) \leq 0$ for $x_T \in [0, 1]$, which implies $x_T^M = 0$.

Now suppose that $x_T^H > 0$. For $j \notin Supp(\Gamma_H)$, we have

$$\hat{f}(x_T^H) - \gamma_j t v(x_T^H) \leq 0 \implies \hat{f}(x_T^H) - \gamma t v(x_T^H) \leq 0,$$

for every $\gamma \geq \gamma_j$. Therefore, $Supp(\Gamma_H)$ consists of a set of players with smallest sensitivity parameters. Since $x_T^H > 0$, player $1 \in Supp(\Gamma_H)$. From equation (7) for Γ_H, we have

$$\begin{aligned} x_T^H &= \sum_{i \in Supp(\Gamma_H)} g_i(x_T^H, t) = \sum_{i=1}^{n} \max(g_i(x_T^H, t), 0) \\ &= \sum_{i=1}^{n} \max\left(\frac{\hat{f}(x_T^H) - \gamma_i t v(x_T^H)}{-\hat{f}'(x_T^H) + \gamma_i t v'(x_T^H)}, 0 \right) \\ &\triangleq \sum_{i=1}^{n} \max(h_{x_T^H,t}(\gamma_i), 0), \end{aligned}$$

where $h_{x_T^H,t}(\cdot)$ is a function of the tax sensitivity γ at a given total investment x_T^H and tax rate t. Note that, since the players are loss averse, we have $v'(x_T) = (k-1)p'(x_T) > 0$. As a result, for $\gamma \geq \gamma_1$, the numerator of $h_{x_T^H,t}(\gamma)$ is strictly decreasing in γ, while the denominator is strictly increasing in γ. We now define an interval $\mathcal{J} \subseteq [\gamma_1, 1]$ as follows.

If $h_{x_T^H,t}(1) > 0$, then $\mathcal{J} = [\gamma_1, 1]$. Otherwise, $\mathcal{J} = [\gamma_1, \bar{\gamma})$, where $\bar{\gamma} \leq 1$ is the unique tax sensitivity at which $h_{x_T^H,t}(\bar{\gamma}) = 0$, and every player $i \in Supp(\Gamma_H)$ satisfies $\gamma_i \in \mathcal{J}$. For $\gamma \in \mathcal{J}$, we have $\hat{f}(x_T^H) - \gamma t v(x_T^H) > 0$ and $-\hat{f}'(x_T^H) + \gamma t v'(x_T^H) > 0$, which implies

$$\hat{f}(x_T^H)v'(x_T^H) > \gamma t v(x_T^H)v'(x_T^H) > \hat{f}'(x_T^H)v(x_T^H). \quad (14)$$

For $\gamma \in \mathcal{J}$, straightforward calculations yield

$$\begin{aligned} h'_{x_T^H,t}(\gamma) &= \frac{t(\hat{f}'(x_T^H)v(x_T^H) - \hat{f}(x_T^H)v'(x_T^H))}{(-\hat{f}'(x_T^H) + \gamma t v'(x_T^H))^2}, \\ h''_{x_T^H,t}(\gamma) &= \frac{-2v'(x_T^H)t^2(\hat{f}'(x_T^H)v(x_T^H) - \hat{f}(x_T^H)v'(x_T^H))}{(-\hat{f}'(x_T^H) + \gamma t v'(x_T^H))^3}. \end{aligned}$$

Following (14), we have $h''_{x_T^H,t}(\gamma) > 0$ for $\gamma \in \mathcal{J}$. Therefore, $\max(h_{x_T^H,t}(\gamma), 0)$ is continuous and convex for $\gamma \in [\gamma_1, 1]$. Applying Jensen's inequality, we obtain

$$x_T^H = \sum_{i=1}^{n} \max(h_{x_T^H,t}(\gamma_i), 0) \geq n \max(h_{x_T^H,t}(\gamma_m), 0).$$

We now consider two cases. First, suppose $h_{x_T^H,t}(\gamma_m) \leq 0$. Note that $-\hat{f}'(x_T^H) + \gamma_m t v'(x_T^H) > 0$ (since $\gamma_m \geq \gamma_1$ and $v'(x_T^H) > 0$). Thus, we have $\hat{f}(x_T^H) - \gamma_m t v(x_T^H) \leq 0$. When $r(x_T)$ is decreasing, it is easy to see that $\hat{f}(x_T) - \gamma_m t v(x_T) < 0$ for $x_T \in (x_T^H, 1]$. For an increasing and concave $r(x_T)$, $\hat{f}(x_T) - \gamma_m t v(x_T)$ is strictly concave in x_T. Since $\hat{f}'(x_T^H) - \gamma_m t v'(x_T^H) < 0$ and $\hat{f}(x_T^H) - \gamma_m t v(x_T^H) \leq 0$, we have $\hat{f}(x_T) - \gamma_m t v(x_T) < 0$ for $x_T \in (x_T^H, 1]$. Thus, $x_T^M \leq x_T^H$.

Now suppose $h_{x_T^H,t}(\gamma_m) > 0$, i.e., $\hat{f}(x_T^H) - \gamma_m t v(x_T^H) > 0$ and $\hat{f}'(x_T^H) - \gamma_m t v'(x_T^H) < 0$. Assume on the contrary that $x_T^M > x_T^H$. Thus, we have $[x_T^H, x_T^M] \subset \mathcal{I}_m^t$, where $\mathcal{I}_m^t$ is the interval defined in Lemma 1 for a player m with tax sensitivity γ_m. Following Lemma 2, we obtain

$$x_T^H \geq n h_{x_T^H,t}(\gamma_m) = n g_m(x_T^H, t) > n g_m(x_T^M, t) = x_T^M,$$

which is a contradiction. ∎

X. Conclusion

We studied a game-theoretic model of human decision makers competing over a shared failure-prone resource. We investigated the impact of behavioral risk preferences of users on how they respond to a taxation mechanism. We first showed that in resources that exhibit network effects, prospect-theoretic utilities of the players can lead to increase in utilization and fragility with higher tax rates. However, when the players are loss averse or the resource exhibits congestion effects, a higher tax rate leads to a decrease in fragility of the resource. Finally, we showed that for loss averse players, heterogeneity in sensitivities to the tax rate results in larger utilization compared to the case where players have homogeneous tax sensitivities, provided the average sensitivity is identical in both cases. Obtaining an upper bound on the utilization under heterogeneous sensitivities remains an open question.

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