EE595S: Class Lecture Notes
Chapter 3: Reference Frame Theory

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Reference Frame Theory

- Power of Reference Frame Theory:
  - Eliminates Rotor Position Dependence Inductances and Capacitances
  - Transforms Nonlinear Systems to Linear Systems for Certain Cases
  - Fundamental Tool For Rigorous Development of Equivalent Circuits
  - Can Be Used to Make AC Quantities Become DC Quantities
  - Framework of Most Controllers
History of Reference Frame Theory

- 1929: Park’s Transformation
  - Synchronous Machine; Rotor Reference Frame
- 1938: Stanley
  - Induction Machine; Stationary Reference Frame
- 1951: Kron
  - Induction Machine; Synchronous Reference Frame
- 1957: Brereton
  - Induction Machine; Rotor Reference Frame
- 1965: Krause
  - Arbitrary Reference Frame
3.3 Equations of Transformation

- \( f_{qd0s} = K_s f_{abcs} \)  \hspace{1cm} (3.3-1)

- \((f_{qd0s})^T = [f_{qs} \quad f_{ds} \quad f_{0s}]\)  \hspace{1cm} (3.3-2)

- \((f_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}]\)  \hspace{1cm} (3.3-3)
3.3 Equations of Transformation

- **Forward Transformation**
  \[
  K_s = \frac{2}{3} \begin{bmatrix}
    \cos \theta & \cos \left(\theta - \frac{2\pi}{3}\right) & \cos \left(\theta + \frac{2\pi}{3}\right) \\
    \sin \theta & \sin \left(\theta - \frac{2\pi}{3}\right) & \sin \left(\theta + \frac{2\pi}{3}\right) \\
    \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
  \end{bmatrix}
  \]  
  (3.3-4)

- **Inverse Transformation**
  \[
  (K_s)^{-1} = \begin{bmatrix}
    \cos \theta & \sin \theta & 1 \\
    \cos \left(\theta - \frac{2\pi}{3}\right) & \sin \left(\theta - \frac{2\pi}{3}\right) & 1 \\
    \cos \left(\theta + \frac{2\pi}{3}\right) & \sin \left(\theta + \frac{2\pi}{3}\right) & 1
  \end{bmatrix}
  \]  
  (3.3-6)
3.3 Equations of Transformation

- Geometrical Interpretation

Figure 3.3-1 Transformation for stationary circuits portrayed by trigonometric relationships.
3.3 Equations of Transformation

- Transformation of Power
  - Starting Point
    - \( P_{abcs} = v_{as}i_{as} + v_{bs}i_{bs} + v_{cs}i_{cs} \)  \( (3.3-9) \)

  - Result
    - \( P_{qd0s} = P_{abcs} \)  \( (3.3-10) \)
      \[
      = \frac{3}{2}(v_{qs}i_{qs} + v_{ds}i_{ds} + 2v_{0s}i_{0s})
      \]
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

• Resistive Circuits
  \[ v_{abc} = r_s i_{abc} \quad (3.4-1) \]
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

- Inductive Circuits

\[ \mathbf{v}_{abcs} = p\mathbf{\lambda}_{abcs} \] (3.4-4)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

• Inductive Circuits (Continued)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

- Capacitive Circuits

\[ i_{abc} = Pq_{abc} \]  

(3.4-19)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

- Capacitive Circuits (Continued)
3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Example 3B

\[ r_S = \text{diag} \begin{bmatrix} r_S & r_S & r_S \end{bmatrix} \quad (3B-1) \]

\[ L_S = \begin{bmatrix} L_S & M & M \\ M & L_S & M \\ M & M & L_S \end{bmatrix} \quad (3B-2) \]
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

• Example 3B (Continued)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

• Example 3B (Continued)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

• Example 3B: Summary

➢ Voltage Equations
• \( v_{qs} = r_s i_{qs} + \omega \lambda_{ds} + p\lambda_{qs} \) (3B-13)
• \( v_{ds} = r_s i_{ds} - \omega \lambda_{qs} + p\lambda_{ds} \) (3B-14)
• \( v_{0s} = r_s i_{0s} + p\lambda_{0s} \) (3B-15)

➢ Flux Linkage Equations
• \( \lambda_{qs} = (L_s - M)i_{qs} \) (3B-16)
• \( \lambda_{ds} = (L_s - M)i_{ds} \) (3B-17)
• \( \lambda_{0s} = (L_s + 2M)i_{0s} \) (3B-18)
3.4 Stationary Circuit Variables
Transformed to Arbitrary Reference Frame

- Equivalent Circuit
# 3.5 Commonly Used Reference Frames

<table>
<thead>
<tr>
<th>Reference frame speed</th>
<th>Interpretation</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$ (unspecified)</td>
<td>Stationary circuit variables referred to the arbitrary reference frame</td>
<td>$f_{q0}, f_{d0}, f_{0s}$</td>
</tr>
<tr>
<td>0</td>
<td>Stationary circuit variables referred to the stationary reference frame</td>
<td>$f_{q0}, f_{d0}, f_{0s}$</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>Stationary circuit variables referred to a reference frame fixed in the rotor</td>
<td>$f_{q0}, f_{d0}, f_{0s}$</td>
</tr>
<tr>
<td>$\omega_e$</td>
<td>Stationary circuit variables referred to the synchronously rotating reference frame</td>
<td>$f_{q0}, f_{d0}, f_{0s}$</td>
</tr>
</tbody>
</table>
3.6 Transformation Between Reference Frames

- Consider Reference Frame $x$
  \[ f^y_{qd0s} = K^y_s f_{abcs} \]  
  \[ (3.6-1) \]

- Consider Reference Frame $y$
  \[ f^x_{qd0s} = K^x_s f_{abcs} \]  
  \[ (3.6-2) \]

- Thus
3.6 Transformation Between Reference Frames

• Evaluating Yields

\[ x^y K_y = \begin{bmatrix}
\cos(\theta_y - \theta_x) & -\sin(\theta_y - \theta_x) & 0 \\
\sin(\theta_y - \theta_x) & \cos(\theta_y - \theta_x) & 0 \\
0 & 0 & 1
\end{bmatrix} \]  

(3.6-7)

• It Can Be Shown That

\[ (x^y K_y)^{-1} = (x^y K_y)^T \]  

(3.6-8)
3.7 Transformation of a Balanced Set

• Consider 3-phase Variables

\[
\begin{align*}
\sigma_{as} &= \sqrt{2}f_s \cos \theta_{ef} \quad (3.7-1) \\
\sigma_{bs} &= \sqrt{2}f_s \cos \left( \theta_{ef} - \frac{2\pi}{3} \right) \quad (3.7-2) \\
\sigma_{cs} &= \sqrt{2}f_s \cos \left( \theta_{ef} + \frac{2\pi}{3} \right) \quad (3.7-3)
\end{align*}
\]

• Where

\[
\omega_e = \frac{d\theta_{ef}}{dt} \quad (3.7-4)
\]
3.7 Transformation of a Balanced Set

• It Can Be Shown

\[ f_{qs} = \sqrt{2} f_s \cos (\theta_{ef} - \theta) \]  \hspace{1cm} (3.7-5)

\[ f_{ds} = -\sqrt{2} f_s \sin (\theta_{ef} - \theta) \]  \hspace{1cm} (3.7-6)
3.7 Transformation of a Balance Set

• Some Special Cases …
3.8 Balanced Steady-State Phasor Relationships

- For Steady State Conditions…
  
  \[ F_{as} = \sqrt{2} F_s \cos[\omega_c t + \theta_{ef}(0)] \]  
  \[ = \text{Re}[\sqrt{2} F_s e^{j\theta_{ef}(0)} e^{j\omega_c t}] \]  
  (3.8-1)

  \[ F_{bs} = \sqrt{2} F_s \cos[\omega_c t + \theta_{ef}(0) - \frac{2\pi}{3}] \]  
  \[ = \text{Re}[\sqrt{2} F_s e^{j[\theta_{ef}(0)-2\pi/3]} e^{j\omega_c t}] \]  
  (3.8-2)

  \[ F_{cs} = \sqrt{2} F_s \cos[\omega_c t + \theta_{ef}(0) + \frac{2\pi}{3}] \]  
  \[ = \text{Re}[\sqrt{2} F_s e^{j[\theta_{ef}(0)+2\pi/3]} e^{j\omega_c t}] \]  
  (3.8-3)
3.8 Balanced Steady-State Phasor Relationships

- From (3.7-5) and (3.7-6)

\[ F_{qs} = \sqrt{2} F_s \cos[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)] \quad (3.8-4) \]

\[ = \text{Re}[\sqrt{2} F_s e^{j[\theta_{ef}(0) - \theta(0)]} e^{j(\omega_e - \omega)t}] \]

\[ F_{ds} = -\sqrt{2} F_s \sin[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)] \quad (3.8-5) \]

\[ = \text{Re}[j\sqrt{2} F_s e^{j[\theta_{ef}(0) - \theta(0)]} e^{j(\omega_e - \omega)t}] \]
3.8 Balanced Steady-State Phasor Relationships

• From (3.8-1)
  \[ \tilde{F}_{as} = F_s e^{j\theta_{ef}(0)} \]  
  \[(3.8-6)\]

• Consider Non-Synchronous Reference Frame
  \[ \tilde{F}_{qs} = F_s e^{j[\theta_{ef}(0) - \theta(0)]} \]  
  \[(3.8-7)\]
  \[ \tilde{F}_{ds} = j\tilde{F}_{qs} \]  
  \[(3.8-8)\]

• With Appropriate Choice of Reference Frame
  \[ \tilde{F}_{as} = \tilde{F}_{qs} \]  
  \[(3.8-9)\]
3.8 Balanced Steady-State Phasor Relationships

• For Synchronous Reference Frame

\[ F_{qs}^e = \text{Re}[\sqrt{2}F_s e^{j[\theta_{ef}(0) - \theta_e(0)]}] \]  

(3.8-10)

\[ F_{ds}^e = \text{Re}[j\sqrt{2}F_s e^{j[\theta_{ef}(0) - \theta_e(0)]}] \]  

(3.8-11)

• With Appropriate Choice of Frame

\[ F_{qs}^e = \sqrt{2}F_s \cos \theta_{ef}(0) \]  

(3.8-12)

\[ F_{ds}^e = -\sqrt{2}F_s \sin \theta_{ef}(0) \]  

(3.8-13)
3.8 Balanced Steady-State Phasor Relationships

• Thus, Choosing Time Zero Position of Zero

\[ \sqrt{2}\tilde{F}_{as} = F_{qs}^e - jF_{ds}^e \]  

(3.8-14)
3.9 Balanced Steady-State Voltage Equations

• Consider Non-Synchronous Reference Frame

• Recall

  \[ \tilde{F}_{as} = F_s e^{j\theta_{ef}(0)} \] (3.8-6)

  \[ \tilde{F}_{qs} = F_s e^{j[\theta_{ef}(0) - \theta(0)]} \] (3.8-7)

• Now Suppose

  \[ \tilde{V}_{as} = Z_s \tilde{I}_{as} \] (3.9-4)

• We Can Show That

  \[ \tilde{V}_{qs} = Z_s \tilde{I}_{qs} \] (3.9-9)
3.9 Balanced Steady-State Voltage Equations

• Proof
3.10 Variables Observed From Several Frames of Reference

• Suppose Following Voltages Applied to 3-Phase Wye-Connected RL Circuit

\[ v_{as} = \sqrt{2}V_s \cos \omega_e t \] (3.10-1)

\[ v_{bs} = \sqrt{2}V_s \cos \left( \omega_e t - \frac{2\pi}{3} \right) \] (3.10-2)

\[ v_{cs} = \sqrt{2}V_s \cos \left( \omega_e t + \frac{2\pi}{3} \right) \] (3.10-3)
3.10 Variables Observed From Several Frames of Reference

- Using Basic Circuit Analysis Techniques

\[
\begin{align*}
\mathbf{i}_{as} &= \frac{\sqrt{2V_s}}{|Z_s|} \left[ -e^{-t/\tau} \cos \alpha + \cos (\omega_e t - \alpha) \right] \quad (3.10-4) \\
\mathbf{i}_{bs} &= \frac{\sqrt{2V_s}}{|Z_s|} \left[ -e^{-t/\tau} \cos \left( \alpha + \frac{2\pi}{3} \right) + \cos \left( \omega_e t - \alpha - \frac{2\pi}{3} \right) \right] (3.10-5) \\
\mathbf{i}_{cs} &= \frac{\sqrt{2V_s}}{|Z_s|} \left[ -e^{-t/\tau} \cos \left( \alpha - \frac{2\pi}{3} \right) + \cos \left( \omega_e t - \alpha + \frac{2\pi}{3} \right) \right] (3.10-6)
\end{align*}
\]
3.10 Variables Observed From Several Frames of Reference

- In (3.10-4)-(3.10-8)

\[ Z_s = r_s + j \omega_e L_s \]  \hspace{1cm} (3.10-7)

\[ \tau = \frac{L_s}{r_s} \]  \hspace{1cm} (3.10-8)

\[ \alpha = \tan^{-1} \left( \frac{\omega_e L_s}{r_s} \right) \]  \hspace{1cm} (3.10-9)
3.10 Variables Observed From Several Frames of Reference

- Transforming to the Synchronous Reference Frame

\[ i_{qs} = \frac{\sqrt{2V_s}}{|Z_s|} \left\{ -e^{-t/\tau} \cos(\omega t - \alpha) \right\} \]

\[ + \cos[(\omega_e - \omega)t - \alpha] \]

\[ i_{ds} = \frac{\sqrt{2V_s}}{|Z_s|} \left\{ -e^{-t/\tau} \sin(\omega t - \alpha) \right\} \]

\[ - \sin[(\omega_e - \omega)t - \alpha] \} \]
3.10 Variables Observed From Several Frames of Reference

- In Stationary Reference Frame

![Graphs of variables](image)

**Figure 3.10-1** Variables of a stationary 3-phase system in the stationary reference frame.
3.10 Variables Observed From Several Frames of Reference

- In Synchronous Reference Frame

![Graph showing variables in synchronous reference frame](image)

**Figure 3.10-2** Variables of a stationary 3-phase system in synchronously rotating reference frame.
3.10 Variables Observed From Several Frames of Reference

- In Strange Reference Frame

![Diagram showing variables](image)

**Figure 3.10-3** Variables of a stationary 3-phase system. First with $\omega = -\omega_e$, then $\omega$ is stepped to zero followed by a ramp change in reference frame speed to $\omega = \omega_e$. 
Transformation of Measured Quantities
Transformation of Measured Currents

• Suppose we are measuring 2 of 3 currents in a wye-connected load …
Transformation of Measured Currents

• Result

\[ K_i^e = \frac{2}{\sqrt{3}} \begin{bmatrix} \cos(\theta_e - \pi / 6) & \sin(\theta_e) \\ \sin(\theta_e - \pi / 6) & -\cos(\theta_e) \end{bmatrix} \]
Transformation of Measured Line-to-Line Voltages

- Suppose we are measuring voltages in a delta-connected load …
Transformation of Line-to-Line Voltages

• Result

\[
K^e_v = \frac{2}{3} \begin{bmatrix}
\cos(\theta_e) & - \cos(\theta_e + 2\pi/3) \\
\sin(\theta_e) & - \sin(\theta_e + 2\pi/3)
\end{bmatrix}
\]