EE595S: Class Lecture Notes
Chapter 3: Reference Frame Theory

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Reference Frame Theory

- Power of Reference Frame Theory:
  - Eliminates Rotor Position Dependence Inductances and Capacitances
  - Transforms Nonlinear Systems to Linear Systems for Certain Cases
  - Fundamental Tool For Rigorous Development of Equivalent Circuits
  - Can Be Used to Make AC Quantities Become DC Quantities
  - Framework of Most Controllers

History of Reference Frame Theory

- 1929: Park's Transformation
  - Synchronous Machine; Rotor Reference Frame
- 1938: Stanley
  - Induction Machine; Stationary Reference Frame
- 1951: Kron
  - Induction Machine; Synchronous Reference Frame
- 1957: Brereton
  - Induction Machine; Rotor Reference Frame
- 1965: Krause
  - Arbitrary Reference Frame

3.3 Equations of Transformation

- $\mathbf{f}_{qd0s} = \mathbf{K}_s \, \mathbf{f}_{abcs}$ (3.3-1)

- $(\mathbf{f}_{qd0s})^T = [\, f_{qs} \quad f_{ds} \quad f_{0s} \,]$ (3.3-2)

- $(\mathbf{f}_{abcs})^T = [\, f_{as} \quad f_{bs} \quad f_{cs} \,]$ (3.3-3)

3.3 Equations of Transformation

- Forward Transformation

$$\mathbf{K}_s = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin\theta & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (3.3\text{-}4)$$

- Inverse Transformation

$$(\mathbf{K}_S)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 1 \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{2\pi}{3}\right) & 1 \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) & 1 \end{bmatrix} \qquad (3.3\text{-}6)$$

3.3 Equations of Transformation

- Geometrical Interpretation

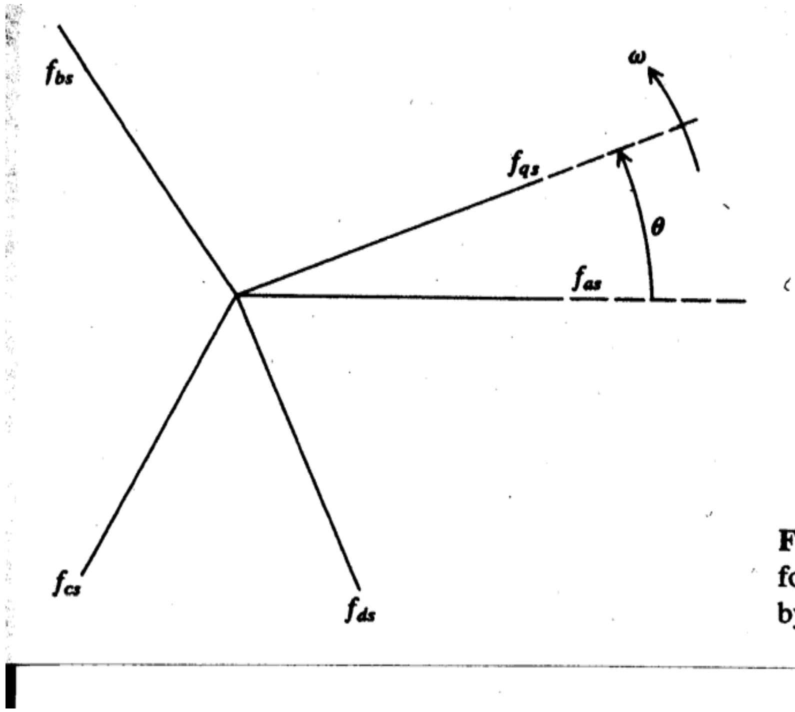


Figure 3.3-1 Transformation for stationary circuits portrayed by trigonometric relationships.

3.3 Equations of Transformation

- Transformation of Power
  - Starting Point
    - $P_{abcs} = v_{as} i_{as} + v_{bs} i_{bs} + v_{cs} i_{cs}$ (3.3-9)
  - Result
    - $P_{qd0s} = P_{abcs}$ (3.3-10)

$$= \frac{3}{2}(v_{qs} i_{qs} + v_{ds} i_{ds} + 2 v_{0s} i_{0s})$$

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Resistive Circuits
  - $\mathbf{v}_{abcs} = \mathbf{r}_s \mathbf{i}_{abcs}$ (3.4-1)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Inductive Circuits
  - $\mathbf{v}_{abcs} = p\boldsymbol{\lambda}_{abcs}$ (3.4-4)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Inductive Circuits (Continued)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

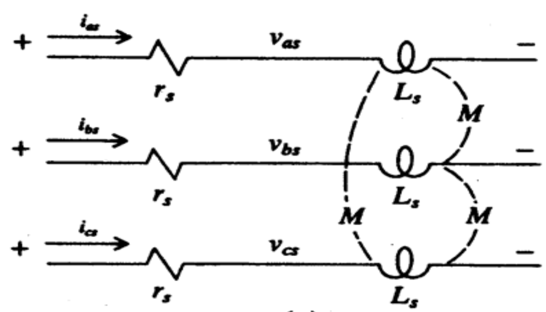
- Capacitive Circuits
  - $\mathbf{i}_{abcs} = p\mathbf{q}_{abcs}$ (3.4-19)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

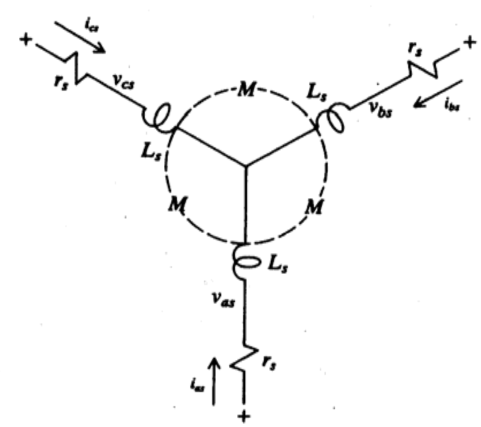
- Capacitive Circuits (Continued)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Example 3B
  - $\mathbf{r}_s = \text{diag}\left[r_s \quad r_s \quad r_s\right]$ (3B-1)
  - $\mathbf{L}_s = \begin{bmatrix} L_s & M & M \\ M & L_s & M \\ M & M & L_s \end{bmatrix}$ (3B-2)



(a)



(b)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Example 3B (Continued)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

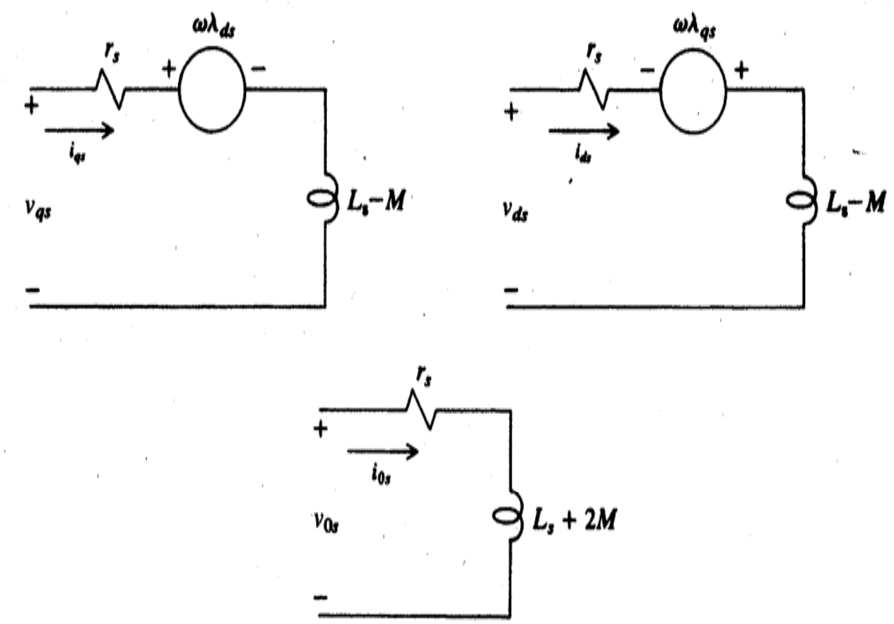
- Example 3B (Continued)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Example 3B: Summary
  - Voltage Equations
    - $v_{qs} = r_s i_{qs} + \omega\lambda_{ds} + p\lambda_{qs}$ (3B-13)
    - $v_{ds} = r_s i_{ds} - \omega\lambda_{qs} + p\lambda_{ds}$ (3B-14)
    - $v_{0s} = r_s i_{0s} + p\lambda_{0s}$ (3B-15)
  - Flux Linkage Equations
    - $\lambda_{qs} = (L_s - M) i_{qs}$ (3B-16)
    - $\lambda_{ds} = (L_s - M) i_{ds}$ (3B-17)
    - $\lambda_{0s} = (L_s + 2M) i_{0s}$ (3B-18)

3.4 Stationary Circuit Variables Transformed to Arbitrary Reference Frame

- Equivalent Circuit



3.5 Commonly Used Reference Frames

Reference frame speed	Interpretation	Notation: Variables	Notation: Transformation
ω (unspecified)	Stationary circuit variables referred to the arbitrary reference frame	$\mathbf{f}_{qd0s}$ or f_{qs}, f_{ds}, f_{0s}	$\mathbf{K}_s$
0	Stationary circuit variables referred to the stationary reference frame	$\mathbf{f}^s_{qd0s}$ or $f^s_{qs}, f^s_{ds}, f_{0s}$	$\mathbf{K}^s_s$
ω_r	Stationary circuit variables referred to a reference frame fixed in the rotor	$\mathbf{f}^r_{qd0s}$ or $f^r_{qs}, f^r_{ds}, f_{0s}$	$\mathbf{K}^r_s$
ω_e	Stationary circuit variables referred to the synchronously rotating reference frame	$\mathbf{f}^e_{qd0s}$ or $f^e_{qs}, f^e_{ds}, f_{0s}$	$\mathbf{K}^e_s$

3.6 Transformation Between Reference Frames

- Consider Reference Frame *x*
  - $\mathbf{f}_{qd0s}^{y} = \mathbf{K}_s^y \mathbf{f}_{abcs}$ (3.6-1)
- Consider Reference Frame *y*
  - $\mathbf{f}_{qd0s}^{x} = \mathbf{K}_s^x \mathbf{f}_{abcs}$ (3.6-2)
- Thus

3.6 Transformation Between Reference Frames

- Evaluating Yields
  - $${}^{x}\mathbf{K}^{y} = \begin{bmatrix} \cos(\theta_y - \theta_x) & -\sin(\theta_y - \theta_x) & 0 \\ \sin(\theta_y - \theta_x) & \cos(\theta_y - \theta_x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.6\text{-}7)$$

- It Can Be Shown That
  - $$({}^{x}\mathbf{K}^{y})^{-1} = ({}^{x}\mathbf{K}^{y})^{T} \qquad (3.6\text{-}8)$$

3.7 Transformation of a Balanced Set

- Consider 3-phase Variables
  - $f_{as} = \sqrt{2} f_s \cos\theta_{ef}$ (3.7-1)
  - $f_{bs} = \sqrt{2} f_s \cos\left(\theta_{ef} - \frac{2\pi}{3}\right)$ (3.7-2)
  - $f_{cs} = \sqrt{2} f_s \cos\left(\theta_{ef} + \frac{2\pi}{3}\right)$ (3.7-3)
- Where
  - $\omega_e = \frac{d\theta_{ef}}{dt}$ (3.7-4)

3.7 Transformation of a Balanced Set

- It Can Be Shown
  - $$f_{qs} = \sqrt{2}\, f_s \cos(\theta_{ef} - \theta) \qquad (3.7\text{-}5)$$
  - $$f_{ds} = -\sqrt{2}\, f_s \sin(\theta_{ef} - \theta) \qquad (3.7\text{-}6)$$

3.7 Transformation of a Balance Set

- Some Special Cases …

3.8 Balanced Steady-State Phasor Relationships

- For Steady State Conditions…
  - $$F_{as} = \sqrt{2}F_s \cos[\omega_e t + \theta_{ef}(0)] = \mathrm{Re}[\sqrt{2}F_s e^{j\theta_{ef}(0)} e^{j\omega_e t}] \quad (3.8\text{-}1)$$
  - $$F_{bs} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{ef}(0) - \frac{2\pi}{3}\right] = \mathrm{Re}[\sqrt{2}F_s e^{j[\theta_{ef}(0) - 2\pi/3]} e^{j\omega_e t}] \quad (3.8\text{-}2)$$
  - $$F_{cs} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{ef}(0) + \frac{2\pi}{3}\right] = \mathrm{Re}[\sqrt{2}F_s e^{j[\theta_{ef}(0) + 2\pi/3]} e^{j\omega_e t}] \quad (3.8\text{-}3)$$

3.8 Balanced Steady-State Phasor Relationships

- From (3.7-5) and (3.7-6)
  - $$F_{qs} = \sqrt{2}F_s \cos[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)] \qquad (3.8\text{-}4)$$
    $$= \mathrm{Re}[\sqrt{2}F_s e^{j[\theta_{ef}(0) - \theta(0)]} e^{j(\omega_e - \omega)t}]$$
  - $$F_{ds} = -\sqrt{2}F_s \sin[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)] \qquad (3.8\text{-}5)$$
    $$= \mathrm{Re}[j\sqrt{2}F_s e^{j[\theta_{ef}(0) - \theta(0)]} e^{j(\omega_e - \omega)t}]$$

3.8 Balanced Steady-State Phasor Relationships

- From (3.8-1)
  - $\tilde{F}_{as} = F_s e^{j\theta_{ef}(0)}$ (3.8-6)
- Consider Non-Synchronous Reference Frame
  - $\tilde{F}_{qs} = F_s e^{j[\theta_{ef}(0) - \theta(0)]}$ (3.8-7)
  - $\tilde{F}_{ds} = j\tilde{F}_{qs}$ (3.8-8)
- With Appropriate Choice of Reference Frame
  - $\tilde{F}_{as} = \tilde{F}_{qs}$ (3.8-9)

3.8 Balanced Steady-State Phasor Relationships

- For Synchronous Reference Frame
  - $F_{qs}^{e} = \mathrm{Re}[\sqrt{2}F_{s}e^{j[\theta_{ef}(0)-\theta_{e}(0)]}]$ (3.8-10)
  - $F_{ds}^{e} = \mathrm{Re}[j\sqrt{2}F_{s}e^{j[\theta_{ef}(0)-\theta_{e}(0)]}]$ (3.8-11)
- With Appropriate Choice of Frame
  - $F_{qs}^{e} = \sqrt{2}F_{s}\cos\theta_{ef}(0)$ (3.8-12)
  - $F_{ds}^{e} = -\sqrt{2}F_{s}\sin\theta_{ef}(0)$ (3.8-13)

3.8 Balanced Steady-State Phasor Relationships

- Thus, Choosing Time Zero Position of Zero
  - $\sqrt{2}\tilde{F}_{as} = F_{qs}^{e} - jF_{ds}^{e}$ (3.8-14)

3.9 Balanced Steady-State Voltage Equations

- Consider Non-Synchronous Reference Frame
- Recall
  - $\tilde{F}_{as} = F_s e^{j\theta_{ef}(0)}$ (3.8-6)
  - $\tilde{F}_{qs} = F_s e^{j[\theta_{ef}(0)-\theta(0)]}$ (3.8-7)
- Now Suppose
  - $\tilde{V}_{as} = Z_s \tilde{I}_{as}$ (3.9-4)
- We Can Show That
  - $\tilde{V}_{qs} = Z_s \tilde{I}_{qs}$ (3.9-9)

3.9 Balanced Steady-State Voltage Equations

- Proof

3.10 Variables Observed From Several Frames of Reference

- Suppose Following Voltages Applied to 3-Phase Wye-Connected RL Circuit
  - $v_{as} = \sqrt{2}V_s \cos\omega_e t$ (3.10-1)
  - $v_{bs} = \sqrt{2}V_s \cos\left(\omega_e t - \frac{2\pi}{3}\right)$ (3.10-2)
  - $v_{cs} = \sqrt{2}V_s \cos\left(\omega_e t + \frac{2\pi}{3}\right)$ (3.10-3)

3.10 Variables Observed From Several Frames of Reference

- Using Basic Circuit Analysis Techniques

  - $$i_{as} = \frac{\sqrt{2}V_s}{|Z_s|}[-e^{-t/\tau}\cos\alpha + \cos(\omega_e t - \alpha)] \qquad (3.10\text{-}4)$$
  - $$i_{bs} = \frac{\sqrt{2}V_s}{|Z_s|}\left[-e^{-t/\tau}\cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\omega_e t - \alpha - \frac{2\pi}{3}\right)\right] \qquad (3.10\text{-}5)$$
  - $$i_{cs} = \frac{\sqrt{2}V_s}{|Z_s|}\left[-e^{-t/\tau}\cos\left(\alpha - \frac{2\pi}{3}\right) + \cos\left(\omega_e t - \alpha + \frac{2\pi}{3}\right)\right] \qquad (3.10\text{-}6)$$

3.10 Variables Observed From Several Frames of Reference

- In (3.10-4)-(3.10-8)

  - $Z_s = r_s + j\omega_e L_s$ (3.10-7)

  - $\tau = \dfrac{L_s}{r_s}$ (3.10-8)

  - $\alpha = \tan^{-1}\dfrac{\omega_e L_s}{r_s}$ (3.10-9)

3.10 Variables Observed From Several Frames of Reference

- Transforming to the Synchronous Reference Frame
  - $$i_{qs} = \frac{\sqrt{2}V_s}{|Z_s|}\{-e^{-t/\tau}\cos(\omega t - \alpha) + \cos[(\omega_e - \omega)t - \alpha]\} \quad (3.10\text{-}10)$$
  - $$i_{ds} = \frac{\sqrt{2}V_s}{|Z_s|}\{-e^{-t/\tau}\sin(\omega t - \alpha) - \sin[(\omega_e - \omega)t - \alpha]\} \quad (3.10\text{-}11)$$

3.10 Variables Observed From Several Frames of Reference

- In Stationary Reference Frame

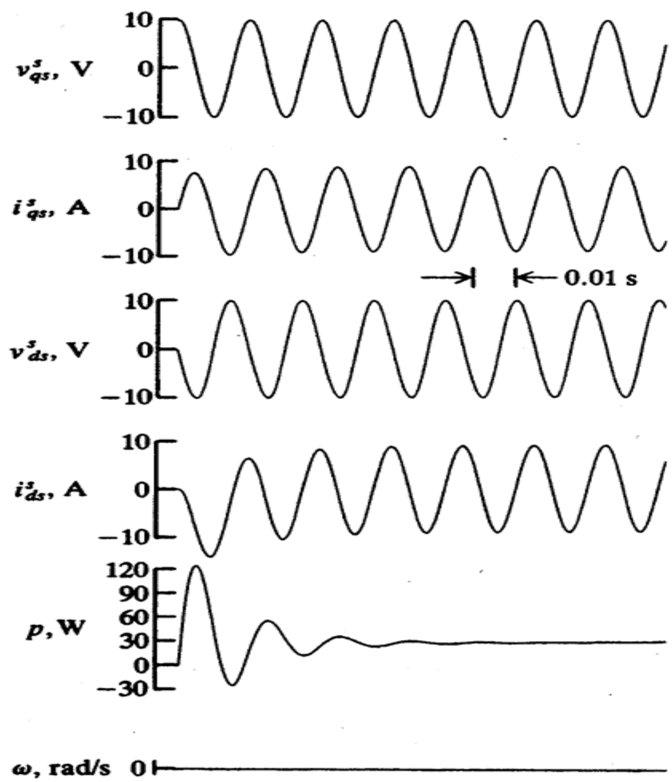


Figure 3.10-1 Variables of a stationary 3-phase system in the stationary reference frame.

3.10 Variables Observed From Several Frames of Reference

- In Synchronous Reference Frame

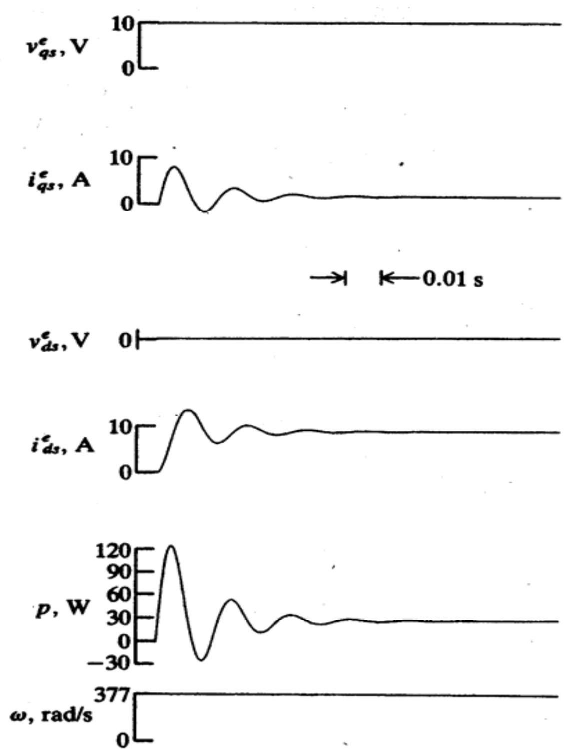


Figure 3.10-2 Variables of a stationary 3-phase system in synchronously rotating reference frame.

3.10 Variables Observed From Several Frames of Reference

- In Strange Reference Frame

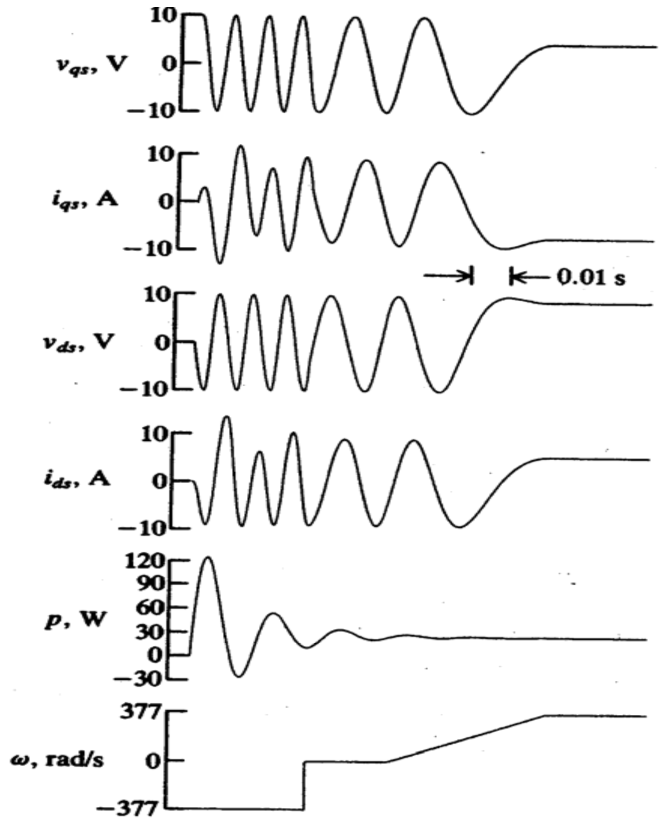
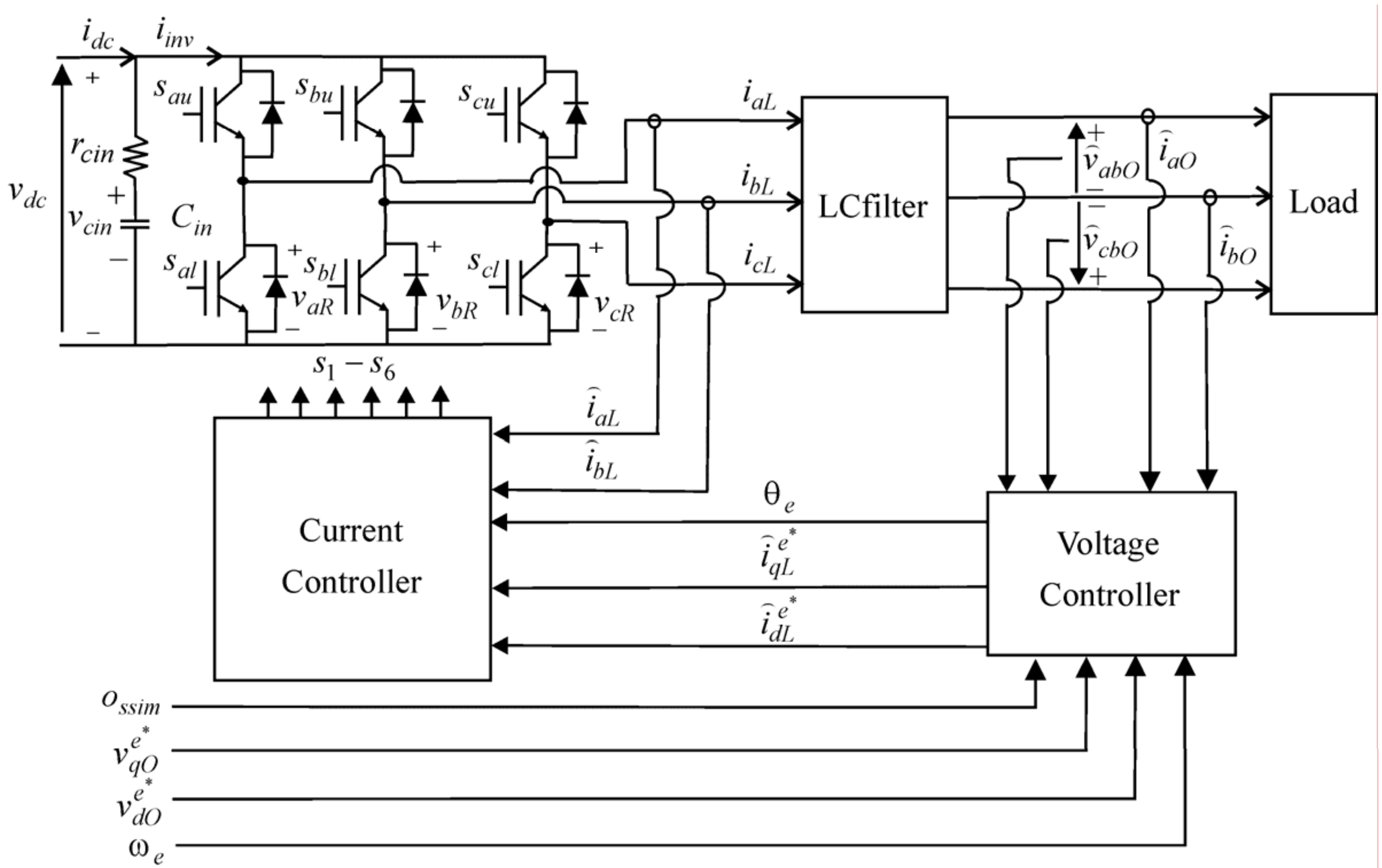


Figure 3.10-3 Variables of a stationary 3-phase system. First with $\omega = -\omega_e$, then ω is stepped to zero followed by a ramp change in reference frame speed to $\omega = \omega_e$.

Transformation of Measured Quantities



Transformation of Measured Currents

- Suppose we are measuring 2 of 3 currents in a wye-connected load …

Transformation of Measured Currents

- Result

$$\mathbf{K}_i^e = \frac{2}{\sqrt{3}}\begin{bmatrix} \cos(\theta_e - \pi/6) & \sin(\theta_e) \\ \sin(\theta_e - \pi/6) & -\cos(\theta_e) \end{bmatrix}$$

Transformation of Measured Line-to-Line Voltages

- Suppose we are measuring voltages in a delta-connected load …

Transformation of Line-to-Line Voltages

- Result

$$\mathbf{K}_v^e = \frac{2}{3}\begin{bmatrix} \cos(\theta_e) & -\cos(\theta_e + 2\pi/3) \\ \sin(\theta_e) & -\sin(\theta_e + 2\pi/3) \end{bmatrix}$$