5/ Force and Torque

In previous chapters, we have concentrated on predicting the electrical aspects of electromagnetic device performance. In this chapter, we consider the production of electromagnetic force and torque. We will consider two approaches to this subject. The first approach will be energy based. Using this approach, given the relationship between flux linkage, current, and position, an algorithm will be set forth to find a corresponding expression for electromagnetic force / torque. The second approach considered will be field based. In this approach, which is geometry-dependent, we will utilize the Lorenz force equation to obtain an expression for torque for a certain class of rotating electromechanical devices.

5.1 AN ENERGY APPROACH TO FORCE AND TORQUE

In this section, an energy based approach to the calculation of force and torque is set forth. In this approach, it is assumed that the relationship between current, flux linkage, and position, i.e. the device flux linkage equation, is known. From this information, a method to derive an expression for force will be set forth. The same method can readily be used to calculate torque.

5.1.1 Magnetic System Description

In order to use the methods set forth in this section, the relationship between flux linkage, current, and mechanical position must be known. It will be assumed herein that this relationship can be expressed in one of two forms. In the first form, the flux linkage may be expressed

$$i = f_{\lambda}(\lambda, x) \tag{5.1-1}$$

where i, λ, and x represent current, flux linkage, and position, respectively. For multi-input systems we have that

$$\mathbf{i} = \mathbf{f}_{\lambda}(\boldsymbol{\lambda}, x) \tag{5.1-2}$$

where the bold font indicates a vector (and flux linkage and current have the same dimension). In (5.1-1) and (5.1-2), $f_{\lambda}()$ is a suitable nonlinear function. The 'λ' subscript is a reminder that the first argument is flux linkage.

The second form of flux linkage equation is used considerably more often than the first and is expressed

$$\lambda = f_i(i, x) \tag{5.1-3}$$

for single-input systems, or, for multi-input systems,

$$\boldsymbol{\lambda} = \mathbf{f}_i(\mathbf{i}, x) \tag{5.1-4}$$

In this case, the 'i' subscript serves as a reminder that the first argument is a current.

5.1.2 Field Energy

Let us now consider an electromechanical device. In general, any such device will involve a magnetic field. The energy stored in this magnetic field will be referred to as the field energy and denoted W_f. Energy that is stored in the magnetic field has two possible sources – the electrical system or the mechanical system. The energy entering the stored field from the electrical system is denoted W_e; energy entering the stored field from the mechanical system is denoted W_m. Assuming that the coupling field is lossless, we have that

$$W_f = W_e + W_m \tag{5.1-5}$$

The assumption of a lossless field does not mean that it is assumed that the device is lossless. Many sources of losses are external to the coupling field (resistive drops, eddy current losses, friction, etc.). However, there is one important source of loss that the assumption of a lossless core does preclude – that of magnetic hysteresis. Nevertheless, the analysis set forth will prove extremely useful and sufficiently accurate in the majority of cases.

As it turns out, the field energy, and a related quantity referred to as the co-energy, will play a critical role in determining force. In order to determine the field energy, let us consider the j'th winding of the electromechanical device. The current into this winding is denoted i_j and the voltage associated with the time rate of flux across the winding is e_j, where

$$e_j = \frac{d\lambda_j}{dt} \tag{5.1-6}$$

The resistive loss across the coil is not considered; this is not because the resistive losses are neglected; rather that they will simply be considered to be external to the energy conversion process.

The energy entering the coupling field through the j'th winding from time t_0 to time t_f may be expressed

$$W_{e,j} = \int_{t_0}^{t_f} e_j i_j \, dt \tag{5.1-7}$$

Substitution of (5.1-6) into (5.1-7) yields

$$W_{e,j} = \int_{t_0}^{t_f} i_j \frac{d\lambda_j}{dt} dt \qquad (5.1\text{-}8)$$

which reduces to

$$W_{e,j} = \int_{\lambda_{j,0}}^{\lambda_{j,f}} i_j \, d\lambda_j \qquad (5.1\text{-}9)$$

where $\lambda_{j,0}$ is λ_j at t_0 and $\lambda_{j,f}$ is λ_j at t_f.

Summing the electrically input energy over all J windings, we have that

$$W_e = \sum_{j=1}^{J} \int_{\lambda_{j,0}}^{\lambda_{j,f}} i_j d\lambda_j \qquad (5.1\text{-}10)$$

Mechanical energy may be expressed as the integral of force over distance; thus we have

$$W_m = -\int_{x_0}^{x_f} f_e dx \qquad (5.1\text{-}11)$$

where f_e is the electromagnetic force, $x = x_0$ at t_0 and $x = x_f$ at $t = t_f$. Therein, it is assumed that the electromagnetic force f_e and displacement x are defined to be in the same direction. The negative sign in (5.1-11) arises from the fact that positive force over a positive distance will cause a positive work being done on the mechanical system, which is a negative contribution to the coupling field.

Substitution of (5.1-10) and (5.1-11) into (5.1-5) yields the primary results of this section; that is that the energy in the coupling field may be expressed

$$W_f = \sum_{j=1}^{J} \int_{\lambda_{j,0}}^{\lambda_{j,f}} i_j d\lambda_j - \int_{x_0}^{x_f} f_e dx \qquad (5.1\text{-}12)$$

5.1.3 Calculation of Field Energy

In the previous section, an expression for the energy in the coupling field was set forth (5.1-12). In this section, we focus on the evaluation of this expression. To this end, we will concentrate on evaluating (5.1-12) when the system flux linkage equations are of the form wherein current is a function of flux linkage and position as in (5.1-1) and (5.1-2). If this is not the case, it is much easier to calculate co-energy; an alternative but equally useful concept which will be introduced in Section 5.1.4.

The field energy represents an energy stored in a lossless and therefore conservative field. As a result, the energy stored in the field is a function of state – and not how that state was reached. In other words, the field energy is a function of present conditions, not past history.

In order to calculate the field energy, we will perform a mathematical experiment. In this experiment, we will specify the trajectory of our state variables (in this case the flux linkage and position) in such a way as to make (5.1-12) as easy to evaluate as possible. Since the field is conservative, the actual trajectory is irrelevant; thus we pick the easiest way to evaluate the trajectory.

To this end, one common trajectory is to pick t_0 to correspond to a point in time wherein the field energy is zero, and the electrical system is unexcited. At this point, since there is no magnetic field, there can be no force due to the magnetic field. Hence, $f_e = 0$. Under these conditions, we position the mechanical system, varying x from its initial value of x_0 at $t = t_0$ to a value of x_f at $t = t_f$. This part of the trajectory does not contribute to W_f since f_e is zero. Thus the expression for field energy reduces from (5.1-12) to

$$W_f = \sum_{j=1}^{J} \int_{\lambda_{j,0}}^{\lambda_{j,f}} i_j d\lambda_j \tag{5.1-13}$$

Substituting our expression for current (5.1-1) or (5.1-2) on an element-by-element basis into (5.1-13) yields

$$W_f = \sum_{j=1}^{J} \int_{\lambda_{j,0}}^{\lambda_{j,f}} f_{i,j}(\boldsymbol{\lambda}, x_f) d\lambda_j \tag{5.1-14}$$

Observe that the '*i*' subscript in (5.1-14) is indicative of the form of the flux linkage equation and is not an index.

The process of evaluating (5.1-14) is best illustrated by a series of examples. In each example, we perform a numerical experiment to bring the system from its' initial state to an arbitrary final state.

Example 5.1.3-1

In this example, consider a single input electrical system wherein the initial current and flux linkage are zero and where

$$i = (5 + 2x)\lambda^2 \tag{5.1-15}$$

Substitution of (5.1-15) into (5.1-14) with x fixed at x_f yields

$$W_f = \frac{1}{3}(5 + 2x_f)\lambda_f^3 \quad (5.1\text{-}16)$$

Since (5.1-16) is valid for any final time, it is convenient to let t_f be the present time t whereupon λ_f is the present value of flux linkage λ, and x_f is the present value of position x. This yields

$$W_f = \frac{1}{3}(5 + 2x)\lambda^3 \quad (5.1\text{-}17)$$

<u>Example 5.1.3-2</u>

Let us now consider a multi-input system, wherein

$$i_1 = 5x\lambda_1 + (10 + 2x)e^{2\lambda_1 + 2\lambda_2} \quad (5.1\text{-}18)$$

$$i_2 = 7\lambda_2 + (10 + 2x)e^{2\lambda_1 + 2\lambda_2} \quad (5.1\text{-}19)$$

Again, we will consider the initial flux linkages to be zero, and the final flux linkages to be $\lambda_1 = \lambda_{1,f}$ and $\lambda_2 = \lambda_{2,f}$. From (5.1-13),

$$W_f = \int_0^{\lambda_{1,f}} i_1 d\lambda_1 + \int_0^{\lambda_{2,f}} i_2 d\lambda_2 \quad (5.1\text{-}20)$$

From evaluating (5.1-20) we will position the mechanical system at $x = x_f$. For this example, it is convenient to observe that the field energy given by (5.1-20) may be expressed as

$$W_f = W_{f,step\,1} + W_{f,step\,2} \quad (5.1\text{-}21)$$

where

$$W_{f,step1} = \int_0^{\lambda_{1,f}} i_1 d\lambda_1 + \int_0^{0} i_2 d\lambda_2 \quad (5.1\text{-}22)$$

and where

$$W_{f,step\,2} = \int_{\lambda_{1,f}}^{\lambda_{1,f}} i_1 d\lambda_1 + \int_0^{\lambda_{2,f}} i_2 d\lambda_2 \quad (5.1\text{-}23)$$

The second term in (5.1-22) and the first term in (5.1-23) are clearly zero. This formulation corresponds to a mathematical experiment wherein in the first step the first step λ_1 is brought from 0 to $\lambda_{1,f}$ with λ_2 fixed at zero. In the second step, λ_2 undergoes a trajectory from 0 to $\lambda_{2,f}$ with λ_1 fixed at $\lambda_{1,f}$. Evaluating (5.1-22) and (5.1-23) yields

$$W_{f,step1} = \frac{5}{2} x_f \lambda_{1,f}^2 + \frac{1}{2}(10 + 2x_f)(e^{2\lambda_1} - 1) \tag{5.1-24}$$

and

$$W_{f,step2} = \frac{7}{2} \lambda_{2,f}^2 + \frac{1}{2}(10 + 2x_f)(e^{2\lambda_{1,f} + 2\lambda_{2,f}} - e^{2\lambda_{1,f}}) \tag{5.1-25}$$

Summing (5.1-24) and (5.1-25) in accordance with (5.1-20) yields

$$W_f = \frac{5}{2} x_f \lambda_{1,f}^2 + \frac{1}{2}(10 + 2x_f)\left(e^{2\lambda_{1,f} + 2\lambda_{2,f}} - 1\right) + \frac{7}{2} \lambda_{2,f}^2 \tag{5.1-26}$$

Since (5.1-26) holds for any final time t_f, we will choose $t_f = t$, whereupon $\lambda_1 = \lambda_{1f}$, $\lambda_2 = \lambda_{2f}$, and $x = x_f$. Thus

$$W_f = \frac{5}{2} x \lambda_1^2 + \frac{1}{2}(10 + 2x)\left(e^{2\lambda_1 + 2\lambda_2} - 1\right) + \frac{7}{2} \lambda_2^2 \tag{5.1-27}$$

Clearly, the process of bringing the flux linkages up one at a time can be extended to systems with any number of inputs. We will consider additional examples of this process when we compute co-energy – a different, but entirely analogous process.

It should be observed that in both of these examples, the field energy was found as a function of position x and the flux linkages. Finding the field energy in terms of the currents can be found in using the considerably more involved procedure set forth, for example, in [1]. However, this process is never really necessary because if the flux linkage equations are in the second form (that is (5.1-3) or (5.1-4)) then it is possible to find the co-energy in a straightforward fashion. The co-energy will be discussed in Section 5.1-4 and is closely related to the field energy.

5.1.4 Calculation of Force as Function of Flux Linkage and Position

Our ultimate objective in Section 5.1 is the calculation of force, not the calculation of field energy. However, as it turns out, once an expression for field energy has been derived an expression for force is readily obtained. To see this, let us first take the total derivative of (5.1-12). By the fundamental theorem of calculus, we have

$$dW_f = \sum_{j=1}^{J} i_j d\lambda_j - f_e dx \qquad (5.1\text{-}28)$$

Now let us suppose, that using the method of Section 5.1.2, we have an expression for the field energy in terms of the flux linkages, $W_f(\boldsymbol{\lambda}, x)$. Taking the total derivative of the field energy yields,

$$dW_f = \sum_{j=1}^{J} \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial \lambda_j} d\lambda_j + \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x} dx \qquad (5.1\text{-}29)$$

Equating (5.1-28) and (5.1-29),

$$\sum_{j=1}^{J} i_j d\lambda_j - f_e dx = \sum_{j=1}^{J} \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial \lambda_j} d\lambda_j + \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x} dx \qquad (5.1\text{-}30)$$

Equation (5.1-30) holds for all infinitesimally small values of $d\lambda_j$ and dx. Since zero is infinitesimally small, let us set all $d\lambda_j$ equal to zero in (5.1-30). This yields the simple and powerful result,

$$f_e = -\frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x} \qquad (5.1\text{-}31)$$

<u>Example 5.1.4-1</u>

In order to demonstrate the utility of (5.1-31), let us reconsider the magnetic system of Example 5.1.2-1, and attempt to calculate the torque. Applying (5.1-31) to (5.1-17) we have

$$f_e = -\frac{2}{3}\lambda^3 \qquad (5.1\text{-}32)$$

<u>Example 5.1.4-2</u>

Let us reconsider the magnetic system set forth in Example 5.1.3-2. In particular, applying (5.1-31) to (5.1-27)

$$f_e = -\left(e^{2\lambda_1 + 2\lambda_2} - 1\right) + \frac{5}{2}\lambda_1^2 \qquad (5.1\text{-}33)$$

Before concluding this section, the importance of the result should again be contemplated. In particular, using the methods of field energy, once an expression for the flux linkages is found, it is straightforward to first find the field energy and then an expression for force. In other words, the flux linkage equations are sufficient information to derive the expression for force. No other additional information (such as geometry, etc.) is needed.

5.1.5 Co-Energy

In the Sections 5.1.3 – 5.1.4, it was assumed that the flux linkage equations were in the form of (5.1-1) or (5.1-2); in particular it was assumed that the currents could be expressed as an explicit function of flux linkage and position. In this section, we consider the more common case wherein flux linkage is expressed as a function of current and position. This corresponds to the formulation set forth in (5.1-3) or (5.1-4).

Our basic approach to finding an expression for torque in this case will be through the use of a concept known as co-energy. To introduce this concept, let us reconsider the evaluation of the field energy given by (5.1-13). In particular, consider the j'th term in the summation which we will define as

$$W_{f,j} = \int_{\lambda_{j,0}}^{\lambda_{j,f}} i_j d\lambda_j \qquad (5.1\text{-}34)$$

Comparing (5.1-13) and (5.1-34), we have

$$W_f = \sum_{j=1}^{J} W_{f,j} \qquad (5.1\text{-}35)$$

The evaluation of (5.1-33) may be viewed geometrically in Figure 5.1-1. Clearly, (5.1-33) corresponds to the indicated area. The corresponding component of co-energy, $W_{c,j}$ is the complementary area also indicated in Figure 5.1-1 and expressed mathematically as

$$W_{c,j} = \int_{i_{j,0}}^{i_{j,f}} \lambda_j \, di_j \qquad (5.1\text{-}36)$$

where $i_{j,0}$ and $i_{j,f}$ correspond to $\lambda_{j,0}$ and $\lambda_{j,f}$, respectively. It is also apparent from Figure 5.1-1 that

$$W_{c,j} + W_{f,j} = \lambda_{f,j} i_{f,j} - \lambda_{0,j} i_{0,j} \qquad (5.1\text{-}37)$$

Throughout Section 5.1, the use of the 'f' subscript on the currents and flux linkages is primarily to allow us to use either current or flux linkage as a variable of integration; since formally we should not use the variable of integration to also be a limit of integration. In other words, we can consider $\lambda_{f,j}$ and $i_{f,j}$ to be the present values of interest λ_j and i_j; we have introduced the 'f' for integral operation just so that we do not have to introduce another dummy variable of integration. For this reason, as we have done earlier, we drop the ‘ f ’ subscript in (5.1-37) which may then be written as

$$W_{c,j} + W_{f,j} = \lambda_j i_j - \lambda_{0,j} i_{0,j} \tag{5.1-38}$$

In order to evaluate $W_{c,j}$ we substitute (5.1-3) or (5.1-4) into (5.1-36) which yields

$$W_{c,j} = \int_{i_{j,0}}^{i_{j,f}} f_{\lambda_j}(\mathbf{i}, x)\, di_j \tag{5.1-39}$$

The total co-energy may then be expressed as

$$W_c = \sum_{j=1}^{J} W_{c,j} \tag{5.1-40}$$

Combining (5.1-39) and (5.1-40),

$$W_c = \sum_{j=1}^{J} \int_{i_{j,0}}^{i_{j,f}} f_{\lambda_j}(\mathbf{i}, x)\, di_j \tag{5.1-41}$$

which is entirely analogous to (5.1-14).

From (5.1-35), (5.1-38), and (5.1-40), it is readily shown that

$$W_c + W_f = \sum_{j=1}^{J} \left(\lambda_j i_j - \lambda_{j,0} i_{j,0} \right) \tag{5.1-42}$$

Equation (5.1-42) has some important physical ramifications. In particular, recall the field energy is a conservative field in that it is only a function of state – not of how that state was achieved. Since the field energy is only a function of state; and the term on the right hand side of the equal sign in (5.1-41) is only a function of state, it follows from (5.1-42) that the co-energy is only a function of state. In other words, the co-energy is a conservative field. Thus (5.1-41) may be evaluated along any trajectory (preferably the one that makes the evaluation easiest) and the same result will be achieved.

<u>Example 5.1.5-1</u>

Let us find the co-energy associated with the following electromechanical system

$$\lambda_1 = 2i_1 + \frac{1}{2+x}(i_1 + i_2)^{0.5} \tag{5.1-43}$$

$$\lambda_2 = 5i_2^{0.4} + \frac{1}{2+x}(i_1 + i_2)^{0.5} \tag{5.1-44}$$

In evaluating the co-energy, we will take an approach similar to that in Example 5.1.3-2. In particular, we will utilize a two step process, wherein in the first step we bring up the first current (in this case from zero) while holding the second current at zero. Then, we hold the first current constant while we bring up the second current.

In particular, starting with (5.1-39) we have

$$W_c = \int_0^{i_{1,f}} \lambda_1 di_1 + \int_0^{i_{2,f}} \lambda_2 di_2 \tag{5.1-45}$$

with x held fixed at x_f, which can be broken up as

$$W_c = W_{c,step1} + W_{c,step2} \tag{5.1-46}$$

where

$$W_{c,step1} = \int_0^{i_{1,f}} \lambda_1\big|_{i_2=0} di_1 + \int_0^{0} \lambda_2 di_2 \tag{5.1-47}$$

and

$$W_{c,step2} = \int_{i_{1,f}}^{i_{1,f}} \lambda_1 di_1 + \int_0^{i_{2,f}} \lambda_2\big|_{i_1=i_{1,f}} di_2 \tag{5.1-48}$$

Clearly, the first term in (5.1-47) and the second term in (5.1-48) are zero. Substitution of (5.1-43) into (5.1-47) with $i_2 = 0$ yields

$$W_{c,step1} = \int_0^{i_{1,f}} 2i_1 + \frac{1}{2+x_f} i_1^{0.5} di_1 \tag{5.1-49}$$

which evaluates to

$$W_{c,step1} = i_{1,f}^2 + \frac{2}{3}\frac{1}{2+x_f} i_{1,f}^{1.5} \tag{5.1-50}$$

For the second step, we bring up the second current while we hold the first current equal to its' final value from step 1. Substituting (5.1-44) into (5.1-48) yields

$$W_{c,step\,2} = \int_0^{i_{2,f}} 5i_2^{0.4} + \frac{1}{2+x_f}(i_{1,f}+i_2)^{0.5}\,di_2 \qquad (5.1\text{-}51)$$

which evaluates to

$$W_{c,step\,2} = \frac{5}{1.4} i_{2,f}^{1.4} + \frac{2}{3}\frac{1}{2+x_f}(i_{1,f}+i_{2,f})^{1.5} - \frac{2}{3}\frac{1}{2+x_f}(i_{1,f})^{1.5} \qquad (5.1\text{-}52)$$

Adding (5.1-50) to (5.1-52) yields

$$W_c = i_{1,f}^2 + \frac{2}{3}\frac{1}{2+x_f}(i_{1,f}+i_{2,f})^{1.5} + \frac{5}{1.4} i_{2,f}^{1.4} \qquad (5.1\text{-}53)$$

As a final note, we observe that t_f, $i_{1,f}$, $i_{2,f}$, and x_f could represent any value of time, current, and position, so we drop the ‘ f ’ subscript leading to our final expression for this example:

$$W_c = i_1^2 + \frac{2}{3}\frac{1}{2+x}(i_1+i_2)^{1.5} + \frac{5}{1.4} i_2^{1.4} \qquad (5.1\text{-}54)$$

<u>Example 5.1.5-2</u>

In this example, we consider the same magnetic system as in Example 5.1.5-1, but we will solve the problem in a different way. In particular, in evaluating (5.1-45) we will assume that the currents follow the trajectory

$$i_1 = i_{1,f}\alpha \qquad (5.1\text{-}55)$$

$$i_2 = i_{2,f}\alpha \qquad (5.1\text{-}56)$$

where α varies from 0 to 1. Observe that from (5.1-55) and (5.1-56) we have that

$$di_1 = i_{1,f}\,d\alpha \qquad (5.1\text{-}57)$$

$$di_2 = i_{2,f}\,d\alpha \qquad (5.1\text{-}58)$$

Incorporating (5.1-55)-(5.1-58) into (5.1-45) we have

$$W_c = \int_0^1 \left(2i_{1,f}\alpha + \frac{1}{2+x}(i_{1,f} + i_{2,f})^{0.5}\alpha^{0.5} \right) i_{1,f}\, d\alpha + \int_0^1 \left(5i_{2,f}^{0.4}\alpha^{0.4} + \frac{1}{2+x}(i_{1,f} + i_{2,f})^{0.5}\alpha^{0.5} \right) i_{2,f}\, d\alpha \tag{5.1-59}$$

which reduces to (5.1-53), whereupon it can be seen that the path of integration did not affect the results, as must be the case for a conservative field.

5.1.6 Calculation of Force as Function of Current and Position

At this point, while the reader may feel comfortable calculating co-energy, the reader may be questioning its use. In answer to this, we will show that the co-energy provides a useful vehicle in the calculation of force.

We begin our development by re-arranging (5.1-42) such that

$$W_c = \sum_{j=1}^{J} \left(\lambda_j i_j - \lambda_{j,0} i_{j,0} \right) - W_f \tag{5.1-60}$$

From which

$$dW_c = \sum_{j=1}^{J} \left(\lambda_j di_j + i_j d\lambda_j \right) - dW_f \tag{5.1-61}$$

Substitution of (5.1-28) into (5.1-61)

$$dW_c = \sum_{j=1}^{J} \left(\lambda_j di_j + i_j d\lambda_j \right) - \sum_{j=1}^{J} i_j d\lambda_j + f_e dx \tag{5.1-62}$$

which reduces to

$$dW_c = \sum_{j=1}^{J} \lambda_j di_j + f_e dx \tag{5.1-63}$$

In our next step, let us suppose that we have the co-energy as a function of current and position, i.e. $W_c(\mathbf{i}, x)$. Taking the total derivative yields,

$$dW_c = \sum_{j=1}^{J} \frac{\partial W_c(\mathbf{i}, x)}{\partial i_j} di_j + \frac{\partial W_f(\mathbf{i}, x)}{\partial x} dx \tag{5.1-64}$$

Equating (5.1-63) and (5.1-64)

$$\sum_{j=1}^{J}\lambda_j di_j + f_e dx = \sum_{j=1}^{J}\frac{\partial W_c(\mathbf{i},x)}{\partial i_j}di_j + \frac{\partial W_f(\mathbf{i},x)}{\partial x}dx \qquad (5.1\text{-}65)$$

Equation (5.1-65) must hold for all infinitesimally small values of di_j and dx. Since zero qualifies as an infinitesimally small value, it is convenient to set $di_j = 0$ for all j. This yields our desired result, namely that

$$f_e = \frac{\partial W_c(\mathbf{i},x)}{\partial x} \qquad (5.1\text{-}66)$$

Equation (5.1-66) is a terribly important result – it is the workhorse result in terms of the calculation of electromagnetic force. It should be committed to memory by anyone interested in electromechanical devices.

Example 5.1.6-1

Let us reconsider the Example 5.1.5-1. In particular, we will find the electromagnetic force for the magnetic system specified therein. Applying (5.1-66) to (5.1-54) yields

$$f_e = -\frac{2}{3}\frac{1}{(2+x)^2}(i_1+i_2)^{1.5} \qquad (5.1\text{-}67)$$

5.1.7 Conditions for Conservative Magnetic Fields

An important assumption of our results in this section are that the field energy and co-energy are conservative fields. We have argued that from a physical viewpoint, this amount to neglecting magnetic hysteresis. However, as it turns out this places a mathematical restriction on how we describe the system flux linkage equations. In particular it can be shown that for a multi-input electrical system of the form (5.1-2) we must have that

$$\frac{\partial \mathbf{f}_{\lambda,j}}{\partial \lambda_k} = \frac{\partial \mathbf{f}_{\lambda,k}}{\partial \lambda_j} \qquad (5.1\text{-}68)$$

and that for a system of the form (5.1-4) we must have

$$\frac{\partial \mathbf{f}_{i,j}}{\partial i_k} = \frac{\partial \mathbf{f}_{i,k}}{\partial i_j} \qquad (5.1\text{-}69)$$

for all $j,k \in [1 \cdots J]$. A discussion of this result as well as the implications of violating these constraints is set forth in [2-4]. The reader is advised to make sure that the flux linkage equations obey (5.1-68) or (5.1-69) as appropriate.

5.1.8 Linear Magnetic Systems

As a useful special case, let us consider the co-energy and field energy produced by a magnetically linear system with J inputs in which

$$\boldsymbol{\lambda} = \mathbf{Li} \tag{5.1-70}$$

Note that from (5.1-69) the $\mathbf{L}$ matrix must be symmetric.

We will first consider the calculation of co-energy. Our initial condition for calculation of the co-energy is that $\mathbf{i} = 0$ and that our final condition is $\mathbf{i} = \mathbf{i}_f$. It is convenient to utilize a trajectory as in Example 5.1.5-2 wherein

$$\mathbf{i} = \alpha \mathbf{i}_f \tag{5.1-71}$$

where α will vary form 0 to 1. Clearly

$$d\mathbf{i}_j = \mathbf{i}_{j,f} d\alpha \tag{5.1-72}$$

Utilizing (5.1-41) in conjunction with (5.1-70)-(5.1-72) we have that

$$W_c = \sum_{j=1}^{J} \int_0^1 \left(\mathbf{Li}_f \alpha \right)\Big|_{j\text{'th row}} i_{j,f} \, d\alpha \tag{5.1-73}$$

which evaluate to

$$W_c = \sum_{j=1}^{J} \frac{1}{2} \left(\mathbf{Li}_f \right)\Big|_{j\text{'th row}} i_{j,f} \tag{5.1-74}$$

Rearranging (5.1-74)

$$W_c = \sum_{j=1}^{J} \frac{1}{2} \left(\mathbf{Li}_f \right)^T \Big|_{j\text{'th column}} i_{j,f} \tag{5.1-75}$$

which may be written more simple as

$$W_c = \frac{1}{2} \mathbf{i}_f^T \mathbf{Li}_f \tag{5.1-76}$$

Dropping the 'f' subscript, the co-energy for a magnetically linear system may be expressed

$$W_c = \frac{1}{2}\mathbf{i}^T\mathbf{L}\mathbf{i} \qquad (5.1\text{-}77)$$

Following an analogous procedure for the field energy, it can be shown that

$$W_f = \frac{1}{2}\boldsymbol{\lambda}^T\mathbf{L}^{-1}\boldsymbol{\lambda} \qquad (5.1\text{-}78)$$

Using the flux linkage equations (5.1-70), it is readily shown that for this linear system the field energy and co-energy are equal, i.e.

$$W_c = W_f \qquad (5.1\text{-}79)$$

This result does not hold for non-linear systems.

5.1.9 Rotational Systems

The expression for mechanical work in a mechanically translational system is the integral of force over a distance as given by (5.1-11). For rotational systems, force is the integral of torque over angular displacement. Thus our expression for mechanical work becomes

$$W_m = -\int_{\theta_{rm,0}}^{\theta_{rm,f}} T_e d\theta_{rm} \qquad (5.1\text{-}80)$$

where T_e is electromagnetic torque and θ_{rm} is rotational position. These quantities are assumed to be defined to be positive in the same direction. Equation (5.1-80) is of the exact same form as (5.1-11) so all of our results for translation systems also hold for rotational systems with the exception that electromagnetic torque T_e replaces force f_e and rotational position θ_{rm} replaces translational position x.

5.1.10 Application to Rotationally Transformable Machinery

As it turns out, for ac electric machinery in which position dependent inductance can be used to eliminate rotor position inductances, there is a short cut approach to the calculation of electric torque. The basic approach will be the same; however the rotor position independence of the flux linkage as expressed in the rotor reference frame will allow us to develop an expression for torque which is valid for any machine in this class.

We will begin our development with (5.1-5). Taking the partial derivative with respect to mechanical rotor position θ_{rm}, we have

$$\frac{\partial W_f}{\partial \theta_{rm}} = \frac{\partial W_e}{\partial \theta_{rm}} + \frac{\partial W_m}{\partial \theta_{rm}} \tag{5.1-81}$$

Our strategy will be to establish expressions for the required partial derivatives and substitute these into (5.1-81). As it turns out, the result will yield an expression for torque.

Our next step is to find an expression of the electrical input energy. Letting t_f be a dummy variable which represents time at the instant of interest, and t_0 be an initial time wherein the system is de-energized, we have that

$$W_e = \int_{t_0}^{t_f} P_e dt \tag{5.1-82}$$

where P_e is the electric power input to the field. The electrical power input may by in turn expressed as

$$P_e = e_{as} i_{as} + e_{bs} i_{bs} + e_{cs} i_{cs} + \sum_{k=1}^{K} e_k i_k \tag{5.1-83}$$

In (5.1-83), i_{as}, i_{bs}, and i_{cs} are the currents into the stator a-, b-, and c-phases, and e_{as}, e_{bs}, and e_{cs} are the time rate of change of the stator phase flux linkages (in other words, the stator phase voltages less the resistive drop). The variables i_k and e_k are the current into and voltage across (less resistive drops) circuits attached to the rotor of the electromechanical device. Denoting the a-, b-, and c-phase flux linkages as λ_{as}, λ_{bs}, and λ_{cs}, and the flux linking the k'th rotor circuit as λ_k, from Faraday's law and (5.1-83) we have

$$P_e = \frac{d\lambda_{as}}{dt} i_{as} + \frac{d\lambda_{bs}}{dt} i_{bs} + \frac{d\lambda_{cs}}{dt} i_{cs} + \sum_{k=1}^{K} \frac{d\lambda_k}{dt} i_k \tag{5.1-84}$$

Transforming the stator quantities to the rotor reference frame using the techniques set forth in Chapter 2, the electrical input power becomes

$$P_e = \frac{3}{2}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right)\omega_r +$$
$$\frac{3}{2}\left(i_{qs}^r \frac{d\lambda_{qs}^r}{dt} + i_{ds}^r \frac{d\lambda_{ds}^r}{dt} + 2i_{0s}\frac{d\lambda_{0s}}{dt}\right) + \tag{5.1-85}$$
$$\sum_{k=1}^{K} \frac{d\lambda_k}{dt} i_k$$

It is convenient to break (5.1-85) into two terms

$$P_e = P_{e1} + P_{e2} \tag{5.1-86}$$

where

$$P_{e1} = \frac{3}{2}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right)\omega_r \tag{5.1-87}$$

and

$$P_{e2} = \frac{3}{2}\left(i_{qs}^r \frac{d\lambda_{qs}^r}{dt} + i_{ds}^r \frac{d\lambda_{ds}^r}{dt} + 2i_{0s}\frac{d\lambda_{0s}}{dt}\right) + \sum_{k=1}^{K} \frac{d\lambda_k}{dt} i_k \tag{5.1-88}$$

It is convenient to break W_e into two corresponding terms such that

$$W_e = W_{e1} + W_{e2} \tag{5.1-89}$$

where

$$W_{e1} = \int_{t_0}^{t_f} \frac{3}{2}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right)\omega_r \, dt \tag{5.1-90}$$

and

$$W_{e2} = \int_{t_0}^{t_f} \left(\frac{3}{2}\left(i_{qs}^r \frac{d\lambda_{qs}^r}{dt} + i_{ds}^r \frac{d\lambda_{ds}^r}{dt} + 2i_{0s}\frac{d\lambda_{0s}}{dt}\right) + \sum_{k=1}^{K} \frac{d\lambda_k}{dt} i_k \right) dt \tag{5.1-91}$$

Let us turn our attention to W_{e1}. Noting

$$\omega_r = \frac{P}{2}\frac{d\theta_{rm}}{dt} \tag{5.1-92}$$

we have that

$$W_{e1} = \frac{3}{2}\frac{P}{2}\int_{t_0}^{t_f}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right)\frac{d\theta_{rm}}{dt}dt \tag{5.1-93}$$

which may be expressed

$$W_{e1} = \frac{3}{2}\frac{P}{2}\int_{\theta_{rm,0}}^{\theta_{rm,f}}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right)d\theta_{rm} \tag{5.1-94}$$

where $\theta_{rm,f}$ is the rotor position at time t_f and $\theta_{rm,0}$ is θ_{rm} at time t_0.

Next, let us consider W_{e2}. From (5.1-91), coupled with the fact that in the rotor reference frame the flux-linkage equations are rotor position invariant, it is clear that the W_{e2} is only a function of flux, and will henceforth be denoted $W_{e2}(\boldsymbol{\lambda})$, where the flux linkage vector λ is defined as

$$\boldsymbol{\lambda} = \left[\lambda_{qs} \quad \lambda_{ds} \quad \lambda_{as} \quad \lambda_1 \quad \cdots \quad \lambda_K\right]^T \tag{5.1-95}$$

Our next step is to take the partial derivative of W_e with respect to mechanical rotor position. From (5.1-89)

$$\frac{\partial W_e}{\partial \theta_{rm}} = \frac{\partial W_{e1}}{\partial \theta_{rm}} + \frac{\partial W_{e2}}{\partial \theta_{rm}} \tag{5.1-96}$$

Using (5.1-94) for the first term, and noting that the second term is zero (since W_{e2} is not a function of rotor position), we have

$$\frac{\partial W_e}{\partial \theta_{rm}} = \frac{3}{2}\frac{P}{2}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right) \tag{5.1-97}$$

We will next address the calculation of the field energy. Since the field energy is a conservative field, we can use (5.1-5) evaluated over any trajectory. As in our earlier work, we will position the mechanical system and then bring up the electrical system. Using such a strategy, the mechanical contribution to the field energy is zero and so we have

$$W_f = W_e\Big|_{\text{Mechanical Position Fixed}} \tag{5.1-98}$$

Observe that if the mechanical rotor position is fixed, W_{e1} is zero. Thus we have

$$W_f = W_{e2} \tag{5.1-99}$$

However, recall that W_{e2} is not a function of position – and thus neither will W_f. It follows that

$$\frac{\partial W_f(\boldsymbol{\lambda})}{\partial \theta_{rm}} = 0 \tag{5.1-100}$$

Our last energy to consider is the mechanical energy W_m. From (5.1-11) (with T_e replacing f_e and θ_{rm} replacing x, as discussed in Section 5.1-9), we have that

$$\frac{\partial W_m}{\partial \theta_{rm}} = -T_e \tag{5.1-101}$$

Substitution of (5.1-97), (5.1-100), and (5.1-101) into (5.1-81) we have that

$$T_e = \frac{3}{2}\frac{P}{2}\left(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r\right) \tag{5.1-102}$$

Transforming (5.1-102) to an arbitrary reference frame, we have our final result

$$T_e = \frac{3}{2}\frac{P}{2}\left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\right) \tag{5.1-103}$$

Equation (5.1-103) is an extremely useful and important result. It gives us an easy-to-use expression for electromagnetic torque for any three-phase machine in which the flux-linkage equations can be made rotor-position-invariant by transformation to the rotor reference frame. The expression is commonly applied to synchronous machines (including certain classes of permanent magnet machines), a variety of reluctance machines, and induction machines. It is highly useful in that it saves the effort of formally computing either the field or co-energy. A similar procedure can also be carried out for two-phase machines; in this case the corresponding result is

$$T_e = \frac{P}{2}\left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\right) \tag{5.1-104}$$

5.2 A Field Approach to Torque Revisited

Equation (5.1-104) is an extremely useful result. However, the derivation assumed that the coupling field is conservative, which means, strictly speaking, that the method breaks down in materials in which exhibit magnetic hysteresis. In this section, an alternate method of deriving electromagnetic torque is derived. It is valid for any distributed winding machine and is valid in the presence of hysteresis. However, there is a disadvantage in that one additional assumption will be made, which is that the slot/winding structure used for practical construction of the stator windings will produce the same amount of torque in the truly continuous distributed winding that it attempts to approximate. Clearly, slot induced torque ripple will not be captured by this method; however the same can be said of the derivation in Section 5.1.10 wherein the flux-linkage equations as expressed in a rotor reference frame were assumed to be rotor position invariant.

5.2.1 Mathematical Development

In this section, three phase electric machinery with distributed windings will be considered. The radius of the location of the stator windings and length of the machine will be denoted r and L, respectively. The turns density of the x-phase winding will be denoted $N_{xs}(\phi_s)$ where '*x*' may be '*a*', '*b*', or '*c*' and ϕ_s denotes the position along the stator as measured from the a-phase axis and proceeding in the counterclockwise direction facing the front of the machine. The turns density is such that turns out of the cross sectional diagram of the machine facing the front of the machine are considered positive.

It is convenient to denote the winding function of each phase as the number of times the winding for that phase spans flux in the direction of rotor to stator at a particular point. The winding function of each phase is denoted $W_{xs}(\phi_s)$ and satisfies

$$\frac{dW_{xs}(\phi_s)}{d\phi_s} = -N_{xs}(\phi_s) \tag{5.2-1}$$

Thus, W_{xs} may be found simply by integrating N_{xs} to within a constant, where the constant may be determined by noting that

$$\int_0^{2\pi} W_{xs}(\phi_s)\,d\phi_s = 0 \tag{5.2-2}$$

Equation (5.2-2) comes about because for every conductor in place in a given direction there will be a conductor in the opposite direction. In terms of the winding function, the magnetizing flux linking the *x*-phase may be expressed

$$\lambda_{xm} = rL\int_0^{2\pi} W_{xs}(\phi_s)B_r(\phi_s)d\phi_s \tag{5.2-3}$$

An expression for torque production may be obtained starting with the Lorenz force equation, which states that the force acting on a single conductor in a machine may be expressed

$$F = iLB_r(\phi_s) \tag{5.2-4}$$

where i is the current out of the page, $B_r(\phi_s)$ is the flux density referenced such that flux flowing from the rotor to the stator is positive, and F is the force which will be at right angles to the conductor and at right angles to the radial flux density and will be in the counter clockwise direction relative to a line drawn from the center of the machine to the conductor.

From (5.2-4), the total torque on the stator in the clockwise direction may be expressed as

$$T_{es} = rL\int_0^{2\pi} B_r(\phi_s)N_{abcs}(\phi_s)^T i_{abcs}d\phi_s \tag{5.2-5}$$

where

$$N_{abcs}(\phi_s) = \begin{bmatrix} N_{as}(\phi_s) & N_{bs}(\phi_s) & N_{cs}(\phi_s) \end{bmatrix}^T \tag{5.2-6}$$

and

$$i_{abcs} = \begin{bmatrix} i_{as} & i_{bs} & i_{cs} \end{bmatrix}^T . \tag{5.2-7}$$

This must also be the negative of the torque on the rotor in the counter-clockwise direction, denoted T_e; i.e.

$$T_e = -T_{es} \tag{5.2-8}$$

For certain winding distributions, including sinusoidal, the turns density may be expressed as a linear function of the winding function as in (5.2-9).

$$\mathbf{N}_{abcs}(\phi_s) = \frac{P}{2}\left[\mathbf{A}\mathbf{W}_{abc}(\phi_s) + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T F_3(\phi_s)\right] \tag{5.2-9}$$

where P is the number of poles, **A** is a constant matrix, and

$$\mathbf{W}_{abcs}(\phi_s) = \begin{bmatrix} W_{as}(\phi_s) & W_{bs}(\phi_s) & W_{cs}(\phi_s) \end{bmatrix}^T , \tag{5.2-10}$$

and $F_3(\phi_s)$ is an arbitrary scalar function of stator position. The purpose of introducing $F_3(\phi_s)$ into (5.2-9) is as artifact which will enable the derivation to handle the case wherein the machine turns density contains a sinusoidal fundamental component plus a considerable triple N harmonic content – the usual case. Substitution of (5.2-8)-(5.2-9) into (5.2-5) yields

$$T_e = -rL\frac{P}{2}\int_0^{2\pi} B_r(\phi_s)\begin{pmatrix} \mathbf{A}\mathbf{W}_{abcs}(\phi_s) + \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T F_3(\phi) \end{pmatrix}^T \mathbf{i}_{abcs} d\phi_s \tag{5.2-11}$$

Assuming that the machine is wye-connected whereupon the zero sequence current is zero, the contribution of the $F_3(\phi_s)$ term in (5.2-11) is zero. Next, comparing (5.2-11) to (5.2-3), the torque may be expressed

$$T_e = -\frac{P}{2}\lambda_{abcs}^T \mathbf{A}^T \mathbf{i}_{abcs} \tag{5.2-12}$$

This is an interesting result in that it holds in the presence of magnetic saturation (as did (5.1-104)) and hysteresis (wherein the derivation of (5.1-104) breaks down).

5.2.2 Sinusoidally Distributed Machines

As a special case, it is useful to consider a quasi-sinusoidally distributed machine in which

$$\mathbf{N}_{abcs}(\phi_s) = N_{pk}\begin{bmatrix} \sin(P\phi_s/2) \\ \sin(P\phi_s/2 - 2\pi/3) \\ \sin(P\phi_s/2 + 2\pi/3) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} N_3(3\phi_s) \tag{5.2-13}$$

where N_{pk} is the peak turns density of the fundamental component of the winding distribution and the $N_3(3\phi_s)$ term represent triplen harmonics in the winding function. This is a good representation of a number of practical machines. Applying (5.2-1)-(5.2-2) to (5.2-13)

$$\mathbf{W}_{abcs}(\phi_s) = (2N_{pk}/P)\begin{bmatrix} \cos(P\phi_s/2) \\ \cos(P\phi_s/2 - 2\pi/3) \\ \cos(P\phi_s/2 + 2\pi/3) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} W_3(3\phi_s) \tag{5.2-14}$$

where the $W_3(3\phi_s)$ term arises from the application of (5.2-1)-(5.2-2) to the $N_3(3\phi_s)$ term in (5.2-13). Comparing (5.2-9), (5.2-13), and (5.2-14) it is apparent that for this winding distribution **A** is given by

$$\mathbf{A} = \frac{\sqrt{3}}{6}\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \tag{5.2-15}$$

In this type of machine, it is often convenient to work in terms of *qd0* variables. Transforming (5.2-12) to the arbitrary reference using the transformation defined by (2.X-X) and (2.X-X)

$$T_e = -\frac{P}{2}\lambda_{qd0m}^T\left[\mathbf{K}_s(\theta)^{-1}\right]^T\mathbf{A}^T\mathbf{K}_s(\theta)^{-1}\mathbf{i}_{qd0s} \tag{5.2-16}$$

which reduces to

$$T_e = \frac{3}{2}\frac{P}{2}\left(\lambda_{dm}i_{qs} - \lambda_{qm}i_{ds}\right) \tag{5.2-17}$$

which is very similar to (5.1-104) except that it is terms of the magnetizing flux linkages. If the common case wherein

$$\lambda_{qs} = \lambda_{qm} + L_{ls}(\bullet)i_{qs} \tag{5.2-18}$$

$$\lambda_{ds} = \lambda_{dm} + L_{ls}(\bullet)i_{ds} \tag{5.2-19}$$

where $L_{ls}(\bullet)$ denotes that the leakage inductance may be a function (of, for example, stator current magnitude or magnetizing flux magnitude), but is identical for the q- and d-axis, then (5.2-17) and (5.1-104) can be readily shown to be entirely equivalent.

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Author's Note (To Himself)

You may want to add problems.