4 / Distributed Windings and Rotating Fields

The objective of this chapter is to develop some analytical tools specifically useful in the analysis of rotational electric machinery. One of the most important properties in this class of machines is that the windings are distributed throughout the machine, rather than concentrated in a particular place, as we have mainly considered thus far. This chapter largely concerns the treatment of distributed windings and how they can be used to create rotating magnetic fields.

In particular, our objectives for this chapter are as follows. First, we need to establish methods to describe distributed windings. There are two methods of doing this – a discrete description and a continuous description. The second objective of this chapter is to set forth a method to determine the flux linked by a distributed winding, and the flux caused by a distributed winding. As it turns out, a concept referred to as a winding function will be very useful in both of these problems. Next, we will turn our attention to the fields caused by distributed windings, and see how, with proper excitation, a rotating field can be established. This rotating field is responsible for the rotation of the rotor. Finally, we will turn our attention to calculating the electrical parameters of distributed winding systems. In particular, we will establish methods to calculate the self inductances, mutual inductances, and resistances of distributed windings.

4.1 NOTATION

Before embarking into the subject of windings and rotating fields, it is convenient to first set forth some notation which will be used throughout this work. In particular, consider Figure 4.1. Therein, a generic electrical machine is depicted. The outside of the machine is stationary and is referred to as the stator. The inner portion of the machine rotates and is referred to as the rotor. In Figure 4.1, half the rotor is shaded so rotor position is evident. Three angles are defined in Figure 4.1. These include

ϕ_{sm} - Position as measured relative to the stator

ϕ_{rm} - Position as measured relative to the rotor

θ_{rm} - The position of the rotor relative to the stator

Clearly, the position of a feature (such as the location of part of a winding) can be described by either ϕ_{sm} or ϕ_{rm}; however, if we are describing the same feature using both of these quantities, then these two measures of angular position are related by

$$\theta_{rm} + \phi_{rm} = \phi_{sm} \tag{4.1-1}$$

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Finally, much of the analysis presented herein may be expressed either in terms of ϕ_{sm} measured relative to the stator or ϕ_{rm} measured relative to the rotor. In this case, we will simply use ϕ to stand for either quantity, as desired.

4.2 DISTRIBUTED WINDINGS

With the notation set forth, we may now proceed to discuss the primary topic of this chapter – distributed windings and rotating fields. We will treat windings in one of two ways – with a discrete description wherein the windings are treated as if they are spread over a finite number of points on the machine and with a continuous description wherein the winding is treated as being continuously distributed. The distinction between the two sets is largely one of analysis – not one of physical difference. Ideally, in 'standard' machines, it is desirable to utilize continuously distributed windings in order to achieve uniform torque. However, for reasons of construction, the wire which makes up the winding are placed into discrete slots in the machine's stator and rotor structures. Thus, it can be said that a discrete winding is used to approximate a continuous one. Yet, the situation is more subtle than this, because if we consider the fact that the winding has physical dimensions, then all windings may be considered to be continuous – and so the real difference becomes one of how we describe the winding mathematically. These concepts will become clear as we proceed in our discussion.

4.2.1 Discrete Windings

Figure 4.2-1 illustrates the stator of a machine in which the stator windings are located in a set of slots (also shown is a path of integration – this will be discussed later in the section). The notation $N_{as,i}$ in Figure 4.2-1 indicates the number of conductors in the i'th slot of the 'a' stator winding. Generalizing this notation, $N_{xy,i}$ is the number of conductors in slot i of winding (also called phase) x of the stator ('y' = 's') or rotor ('y' = 'r'). Often, a slot will contain individuals from more than one winding or phase.

Physically, the location of the center of the i'th stator slot is at a position

$$\phi_{ys,i} = \frac{\pi}{N_{yslt}}(2i-1) \tag{4.2-1}$$

where N_{yslt} is the number of slots (here again 'y' = 's' for stator and 'y' = 'r' for rotor). The iron region between the slots is referred to as the teeth; the center of the i'th tooth is at

$$\phi_{yt,i} = \frac{\pi}{N_{yslt}}(2i-2) \tag{4.2-2}$$

The reader may consider it odd that all the conductors are shown as coming out of the page in Figure 4.2-1. This is to emphasize the definition that conductors will be considered to be positive coming out of the page. Hence $N_{as,1}$, $N_{as,2}$, …, $N_{as,8}$ are all defined as positive out. Physically, however, these quantities can be positive are negative. Since the number of turns going into the page must be equal to the number of conductors out of the page (the wire is formed in closed loops), we have that

$$\sum_{i=1}^{N_{yslt}} N_{xy,i} = 0 \tag{4.2-3}$$

where 'x' may be 'a', 'b', or 'c'. The total number of turns associated with the winding may be expressed

$$N_{xy} = \sum_{i=1}^{N_{yslt}} N_{xy,i}\ u(N_{xy,i}) \tag{4.2-4}$$

where $u(\cdot)$ is one if its argument is positive and zero otherwise.

In the process of describing windings, it will be convenient to use what is commonly referred to as a developed diagram. In the developed diagram, spatial features (such as the location of the winding) are depicted against a linear axis. In essence, the machine becomes 'unrolled'. This process is best illustrated by example; Figure 4.2-2 is the developed diagram corresponding to Figure 4.2-1. One feature which may capture the reader's interest is that the independent axis is directed to the left rather than to the more conventionally choice (to the right). While this can be done, it involves the additional steps of changing the direction of the path of integration as well as the direction of all the windings in the figure, since the figure must be 'flipped' in three-dimensions. To avoid these additional steps, the direction of the independent variable is directed to the left.

4.2.2 Continuous Windings

Machine windings are placed into slots for reasons of physical construction; in most cases it is desirable to continuously distribute the winding in order to provide smooth torque output. In cases where the number of slots is high, it is convenient to represent a winding by the mathematically abstraction of what it is desired to approximate – a truly continuous winding. Continuous windings are described in terms of a turns density, which is a measure of the number of conductors per radian as a function of position. As an example, we would describing winding 'x' of a machine with the turns density $n_{xy}(\phi_{ym})$ where 'x' again denotes the winding (and is commonly 'a', 'b', or 'c'), and where 'y' is 's' for a stator winding or 'r' for a rotor winding. The turns density may be positive or negative; positive turns are considered herein to be out of the page, negative turns are into the page.

It is often the case that the turns density is a sinusoidal function of position. A common choice for a turns density of the a-phase of the stator in three-phase ac machinery is

$$n_{as}(\phi_{sm}) = n_{s1}\cos(P\phi_{sm}/2) - n_{s3}\cos(3P\phi_{sm}/2) \tag{4.2-5}$$

In this function, the first term is the primarily desired one; the second term is used to make the winding easier to implement and to make better use of the machine slots. In (4.2-5), the parameter $P/2$ (which is in the set natural numbers) describes the number of cycles of turns density as we go around the stator. The $P/2$ term is normally referred to as the number of pole pairs, and P as the number of poles. As it turns out, the ratio $P/2$ will largely dictate the relationship between the physical rotational speed of the machine and the frequency of the fundamental component of the stator voltages and currents.

It will often be of interest to determine the total number of turns associated with a winding. This number is readily found by integrating the turns density over all regions of positive (or negative) turns, so that the total number of turns may be expressed

$$N_{xy} = \int_0^{2\pi} n_{xy}(\phi)u(n_{xy}(\phi))d\phi \tag{4.2-6}$$

where, as before, $u(\cdot)$ is one if its argument is positive and zero otherwise.

4.2.3 Conversion of Discrete Description to Continuous Description

Suppose that we have a discrete description of a winding consisting of the number of conductors of each phase in the slots. In functional form, the turns density could be expressed

$$n_{xy}(\phi_{sm}) = \sum_{i=1}^{N_{yt}} N_{xy,i}\delta(\phi_{sm} - \phi_{ys,i}) \tag{4.2-7}$$

where $\delta(\cdot)$ is the unit impulse function. This function is infinite when its argument is zero and zero for all non-zero values of its argument, and has unity area when integrated.

Although (4.2-7) is in a sense a continuous description, normally what we desire is an idealized representation of the desired turns distribution. To this end, we may represent the turns distribution as a single-sided Fourier series of the form

$$n_{xy}(\phi_{sm}) = \sum_{j=1}^{J} a_j\cos(jP\phi_{sm}/2) + b_j\sin(jP\phi_{sm}/2) \tag{4.2-8}$$

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where J is the number of terms used in the series, and where, using the standard expressions for the coefficients, we have

$$a_j = \frac{4\pi}{P} \int_0^{4\pi/P} n_{xy}(\phi_{sm}) \cos(j4\pi\phi_{sm}/P) d\phi_{sm} \tag{4.2-9}$$

$$b_j = \frac{4\pi}{P} \int_0^{4\pi/P} n_{xy}(\phi_{sm}) \sin(j4\pi\phi_{sm}/P) d\phi_{sm} \tag{4.2-10}$$

Substitution of (4.2-7) into (4.2-9) and (4.2-10) yields

$$a_j = \frac{4\pi}{P} \sum_{i=1}^{N_{yslt}} N_{xy,i} \cos(j4\pi\phi_{yt,i}/P) \tag{4.2-11}$$

$$b_j = \frac{4\pi}{P} \sum_{i=1}^{N_{yslt}} N_{xy,i} \sin(j4\pi\phi_{yt,i}/P) \tag{4.2-12}$$

4.2.4 Conversion of Continuous Description to Discrete Description

In this section, it is assumed that we have a continuous winding description, and wish to translate it into a discrete winding description. To this end, we simply add (or integrate, since we are dealing with a continuous function) all conductors within one-half of a slot plus tooth width of the center of the *i*'th slot and lump them into this position. In particular,

$$N_{xy,i} = \text{round}\left(\int_{\phi_{ys,i}-\pi/N_{yslt}}^{\phi_{ys,i}+\pi/N_{yslt}} n_{xy}(\phi_{ym}) d\phi_{ym} \right) \tag{4.2-13}$$

where round() denotes a function which rounds the result to the next nearest integer.

As a brief example, suppose $n_{as} = 100\cos(2\phi_{sm})$. Further suppose that there are to be 24 slots. Using (4.2-8), the number of a-phase conductors in the 8'th slot is -18.

4.2.5 Description of End Turns

Normally, our focus in describing a winding is on the conductors in the axial direction. However, for some purposes, such as calculating winding resistance, it is important to be able to describe the number of conductors on the front and back ends of the machine to connect the slot conductors together.

Herein, we will focus our discussion on a discrete winding description. To this end, define $M_{xy,i}$ as the number of conductors associated with phase x between slots i and $i+1$, where a positive count indicates the windings are in the clockwise direction as viewed from the front of the page. Consideration of the conductor count reveals that

$$M_{xy,i} = M_{xy,i-1} + N_{xy,i} \tag{4.2-14}$$

where the index operations are ring mapped (i.e. $N_{yslt} + 1 \rightarrow 1, 0 \rightarrow N_{yslt}$). From (4.2-14)

$$M_{xy,i} = M_{xy,N_{yslt}} + \sum_{j=1}^{i} N_{xy,j} \tag{4.2-15}$$

In order to utilize (4.2-15), it is necessary to determine $M_{xy,N_{yslt}}$. This can be accomplished by observing that because of symmetry, slot number N_{yslt}/P is diametrically opposed to slot N_{yslt} and so the number of turns is the same but of opposite direction. Thus, we have that

$$M_{xy,N_{yslt}/P} = -M_{xy,N_{yslt}} \tag{4.2-16}$$

Setting $i = N_{yslt}/P$ in (4.2-15), combining the result with (4.2-16), and solving for $M_{xy,N_{yslt}}$ yields

$$M_{xy,N_{yslt}} = -\frac{1}{2}\sum_{j=1}^{N_{yslt}/P} N_{xy,j} \tag{4.2-17}$$

Finally, substitution of (4.2-17) into (4.2-15) yields

$$M_{xy,i} = \sum_{j=1}^{i} N_{xy,j} - \frac{1}{2}\sum_{j=1}^{N_{yslt}/P} N_{xy,j} \tag{4.2-18}$$

The reader should observe that while (4.2-18) expresses our desired result, it is more computationally efficient to use (4.2-18) in conjunction with (4.2-14) since it is normally desired to find $M_{xy,i}$ for all i.

4.3 THE WINDING FUNCTION

Our first goal for this chapter was to identify methods to describe distributed windings. This goal has now been achieved, and we can move on to our second goal – determining the flux linking and caused by a distributed winding. To this end, the first concept that we will need is that of the winding function. The winding function will have three important uses. First, it will be a useful tool in determining the MMF caused by distributed windings. Secondly, it will provide a means to determine how much flux links a winding. Eventually, by using a combination of the first and second of these ideas, the winding function will be instrumental in calculating the self inductance of a winding, as well as the mutual inductance between a winding and another winding.

The winding function is a description of how many times a winding links flux density at any given position. Using this notion will allow us to formulate the mathematical definition of the winding function. To proceed, we must first select a direction for flux to be considered positive. To this end, let us arbitrarily define flux flowing from the rotor to the stator to be positive.

Let us now consider the case in which we use a discrete treatment of the turns function. Figure 4.3-1 illustrates a small portion of the developed diagram of the machine, wherein it is arbitrarily assumed that the winding of interest is on the stator. Let $W_{xy,i}$ denote the number of times winding '*x*' links flux traveling through the *i*'th tooth. Physically, this will serve as our definition of the winding function for a discrete winding description.

Now, let us assume that we know $W_{xy,i}$ for some i. Not so clearly,

$$W_{xy,i+1} = W_{xy,i} - N_{xy,i} \tag{4.3-1}$$

To understand (4.3-1), suppose that $N_{xy,i}$ is positive. Of the $N_{xy,i}$ turns, suppose α of these conductors go to the right (where they turn back into other slots), and β of these conductors go to the left (where again they turn back into other slots). The α turns link flux in tooth i but not flux in tooth $i+1$ since they close the loop to the other side. The β turns which are directed towards the right before closing the loop, do not link tooth i, but do link tooth $i+1$, albeit in the negative direction (which can be seen using the right hand rule and recalling that flux is considered positive from the rotor to the stator). Thus we have that

$$W_{xy,i+1} = W_{xy,i} - \alpha - \beta \tag{4.3-2}$$

Since $N_{xy,i} = \alpha + \beta$, (4.3-2) reduces to (4.3-1).

Manipulation of (4.3-1) yields an expression for the winding function:

$$W_{xy,i} = W_{xy,1} - \sum_{j=1}^{i-1} N_{xy,j} \quad (4.3\text{-}3)$$

in terms of $W_{xy,1}$. In order to establish this term, consider a symmetrically wound machine in which

$$N_{xy,i+N_{yslt}/P} = -N_{xy,i} \quad (4.3\text{-}4)$$

where the indexing operations are ring mapped with a modulus of N_{yslt}. Physically, (4.3-4) states that the number of turns in magnetically diametrically opposed slots are equal in magnitude, but opposite in sign. This is easiest to visualize for a two-pole machine wherein (4.3-4) reduces to a statement that the number of turns going into a slot is the same as the number of turns coming out of the diametrically opposed slot. If (4.3-4) holds, then using (4.3-3) it can be shown that

$$W_{xy,i+N_{yslt}/P} = -W_{xy,i} \quad (4.3\text{-}5)$$

Combining (4.3-3) with $i = 1 + N_{yslt}/P$ and (4.3-5)

$$-W_{xy,1} = W_{xy,1} - \sum_{j=1}^{N_{yslt}/P} N_{xy,j} \quad (4.3\text{-}6)$$

Thus

$$W_{xy,1} = \frac{1}{2}\sum_{j=1}^{N_{yslt}/P} N_{xy,j} \quad (4.3\text{-}7)$$

Although (4.3-7) could be used in conjunction with (4.3-3), it is generally most efficient to find $W_{xy,1}$ using (4.3-7) and then use (4.3-1) to calculate the winding function for each slot in turn. As an additional note, it should be noted that it is assumed that N_{yslt}/P is an integer for the desired symmetry condition to be met. This is not the only restriction on the number of poles and slots. In particular, for a three-phase machine to have three electrically identical phases, and at the same time to ensure that each winding has the needed symmetry, it is required that $N_{yslt}/(3P)$ is an integer.

Let us now consider the calculation of the winding function using a continuous description of the winding. In this case, instead of being a function of the tooth number, the winding function will be a continuous function of position, which can be position relative to the stator $\phi = \phi_{sm}$ for stator windings or position relative to the rotor $\phi = \phi_{rm}$ for rotor windings. Let us assume that we know the winding function at position ϕ, and we wish to calculate the winding function at position $\phi+\Delta\phi$. The number of turns between these two positions is $n_x(\phi)\Delta\phi$ assuming $\Delta\phi$ is small. Using arguments identical to the derivation of (4.3-1) we have that

$$w_{xy}(\phi+\Delta\phi)=w_{xy}(\phi)-n_{xy}(\phi)\Delta\phi \tag{4.3-8}$$

from which

$$\frac{\Delta w_{xy}(\phi)}{\Delta\phi}=-n_{xy}(\phi) \tag{4.3-9}$$

Taking the limit as $\Delta\phi\to 0$,

$$\frac{dw_{xy}}{d\phi}=-n_{xy}(\phi) \tag{4.3-10}$$

Thus the winding function may be calculated as

$$w_{xy}(\phi)=-\int_0^{\phi} n_{xy}(\phi)d\phi+w_{xy}(0) \tag{4.3-11}$$

In order to utilize (4.3-11), we must first establish $w_{xy}(0)$. As in the case of computing $W_{xy,1}$ in the case of a discrete winding description, we establish $w_{xy}(0)$ by requiring that the winding be symmetrical in the sense that

$$n_{xy}(\phi+2\pi/P)=-n_{xy}(\phi) \tag{4.3-12}$$

whereupon by (4.3-10) the winding function must have the same symmetry

$$w_{xy}(\phi+2\pi/P)=-w_{xy}(\phi) \tag{4.3-13}$$

From (4.3-11) with $\phi=2\pi/P$ and using (4.3-13) we have that

$$w_{xy}(0)=\frac{1}{2}\int_0^{2\pi/P} n_{xy}(\phi)d\phi \tag{4.3-14}$$

Finally, substitution of (4.3-14) into (4.3-11) yields

$$w_{xy}(\phi)=\frac{1}{2}\int_0^{2\pi/P} n_{xy}(\phi)d\phi-\int_0^{\phi} n_{xy}(\phi)d\phi \tag{4.3-15}$$

In summary (4.3-1)/(4.3-7) and (4.3-15) provide a means to calculate the winding function for a discrete and continuous winding descriptions, respectfully. The winding function is a physical measure of the number of times a winding links the flux in a particular tooth (discrete winding description) or a particular position (continuous winding description).

Let us now pause in our development to consider two brief examples. First let us consider an example with a discrete winding distribution. To this end, it will be convenient to define the notation.

$$\mathbf{N}_{xy} = \begin{bmatrix} N_{xy,1} & N_{xy,2} & \cdots & N_{xy,N_{yslt}} \end{bmatrix}^T \tag{4.3-16}$$

and

$$\mathbf{W}_{xy} = \begin{bmatrix} W_{xy,1} & W_{xy,2} & \cdots & W_{xy,N_{yslt}} \end{bmatrix}^T \tag{4.3-17}$$

Now let us consider a machine with 12 slots. Suppose,

$$\mathbf{N}_{xy} = \begin{bmatrix} 10 & 20 & 10 & -10 & -20 & -10 & 10 & 20 & 10 & -10 & -20 & -10 \end{bmatrix}^T \tag{4.3-18}$$

and that we wish to calculate the winding function. By inspection of (4.3-18) and comparing it to (4.3-4) it can be seen that this is a 4 pole machine ($P = 4$). From (4.3-7), we have $W_{xy,1} = 20$. From (4.3-1) we then obtain

$$\mathbf{W}_{xy} = \begin{bmatrix} 20 & 10 & -10 & -20 & -10 & 10 & 20 & 10 & -10 & -20 & -10 & 10 \end{bmatrix}^T \tag{4.3-19}$$

We will now consider the calculation of the winding function for a continuous winding description. Suppose we have

$$n_{as} = 100\sin(8\phi_{sm}) \tag{4.3-20}$$

Then, from (4.3-15), we have

$$W_{as} = \frac{100}{8}\cos(8\phi_{sm}) \tag{4.3-21}$$

As it turns out, the winding function will also be very useful in calculating the MMF associated with a winding. The connection between the winding function and the MMF is explored in the next section.

4.4 MAGNETO-MOTIVE FORCE

In this section, we address the production of the MMF across the air gap of the machine. This is important because it is the MMF that causes flux to flow from the rotor to the stator. In doing this, we will concentrate our efforts on the use of a

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continuous winding description, and save our results for a discrete winding description to the consideration of specific machines.

The magneto-motive force (MMF) across the airgap is defined as

$$F(\phi) = \int_{rotor}^{stator} \mathbf{H}(\phi) \cdot d\mathbf{l} \tag{4.4-1}$$

where the path of integration is directed in the radial direction. Because of this, we may rewrite (4.4-1) as

$$F(\phi) = \int_{rotor}^{stator} H(\phi) \cdot dl \tag{4.4-2}$$

where $H(\phi)$ is the radial component of $\mathbf{H}(\phi)$ and $d\mathbf{l}$ reduces to dl since it only has a radial component for the chosen path.

It will be convenient to utilize the principal of superposition to identify the MMF caused by a particular winding. In particular, the airgap MMF associated with winding *x* is expressed

$$F_x(\phi) = \int_{rotor}^{stator} H_x(\phi) \cdot dl \tag{4.4-3}$$

where H_x is that component of radial field intensity caused by winding *x*.

Now let us apply Ampere's law to the path shown in Figures 4.2-1 and 4.2-2. In particular, we have

$$\oint_{abcd} \mathbf{H} \cdot d\mathbf{l} = i_{enc} \tag{4.4-4}$$

Observe that (4.4-4) is identical to requiring that

$$\oint_{abcd} \mathbf{H}_j \cdot d\mathbf{l} = i_{enc,j} \tag{4.4-5}$$

where j denotes the j'th winding, and where

$$\mathbf{H} = \sum_{j=1}^{J} \mathbf{H}_j \tag{4.4-6}$$

$$i_{enc} = \sum_{j=1}^{J} i_{enc,j} \tag{4.4-7}$$

In (4.4-6)-(4.4-7), J is the total number of windings. The significance of approaching the problem in this way is that it will allow us to treat each current source independently.

Now let us consider the MMF due to winding x. Setting $j='x'$ in (4.4-5) we may expand our path integral as

$$\int_a^b \mathbf{H}_x \cdot d\mathbf{l} + \int_b^c \mathbf{H}_x \cdot d\mathbf{l} + \int_c^d \mathbf{H}_x \cdot d\mathbf{l} + \int_d^a \mathbf{H}_x \cdot d\mathbf{l} = i_{enc,x} \tag{4.4-8}$$

Comparing (4.4-8) to (4.4-3), and assuming that the field intensity is negligible in the iron portions of the machine, we have (4.4-3) that

$$F_x(\phi\ ) - F_x(0) = i_{enc,x}(\phi) \tag{4.4-9}$$

where we have now explicitly indicated the functional dependence of the enclosed current on the location of the most counterclockwise edge of the path.

The enclosed current in (4.4-4) may be expressed as the integral of the turns density times the current. In particular, we have that

$$i_{enc,x}(\phi) = -\int_0^\phi n_x(\phi) i_x d\phi \tag{4.4-10}$$

where the minus sign arises from the fact that by virtue of the direction of the path in Figure 4.4-1 positive current is into the page and by our definition of turns density as positive out of the page. Combining (4.4-9) and (4.4-10) we have that

$$F_x(\phi) - F_x(0) = -\int_0^\phi n_x(\phi) i_x d\phi \tag{4.4-11}$$

To proceed further, we need to establish $F_x(0)$. To this, we will make the assumption that the symmetry in the winding function (4.3-11) will give rise to a similar symmetry in the MMF, i.e.

$$F_x(\phi + 2\pi / P) = -F_x(\phi) \tag{4.4-12}$$

whereupon

$$F_x(2\pi / P) = -F_x(0) \tag{4.4-13}$$

Setting $\phi = 2\pi / P$ in (4.4-11) we have that

$$F_x(2\pi / P) - F_x(0) = -\int_0^{2\pi / P} n_x(\phi) i_x d\phi \tag{4.4-14}$$

Combining (4.4-13) and (4.4-14) we arrive at an expression for $F_x(0)$.

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$$F_x(0) = \frac{1}{2}\int_0^{2\pi/P} n_x(\phi) i_x d\phi \qquad (4.4\text{-}15)$$

Our expression for the MMF associate with winding x may be completed by combining (4.4-15) with (4.4-11) and rearranging, which yields

$$F_x(\phi) = \left(\frac{1}{2}\int_0^{2\pi/P} n_x(\phi_{xm}) d\phi_{xm} - \int_0^{\phi_{xm}} n_x(\phi_{xm}) d\phi_{xm} \right) i_x \qquad (4.4\text{-}16)$$

It is interesting, and in fact very important, to compare (4.4-16) to (4.3-15). Doing so, it can be seen that the MMF associated with winding x may be expressed

$$F_x(\phi) = w_x(\phi) i_x \qquad (4.4\text{-}17)$$

This simple result is very important, and illustrates the power of the winding function concept. It can be used to not only calculate how much flux links a winding, but it can also be used to calculate the MMF associated with a winding.

Once we calculate the MMF associated with a winding, we can readily calculate the field intensity and flux density associated with that winding. From (4.4-3), and assuming that the field intensity is constant along the air gap, we have that

$$F_x(\phi) = H_x(\phi) g(\phi) \qquad (4.4\text{-}18)$$

whereupon

$$H_x(\phi) = \frac{F_x(\phi)}{g(\phi)} \qquad (4.4\text{-}19)$$

Since $B = \mu_0 H$ in the air gap, the component of the flux density due to winding x may be expressed

$$B_x(\phi) = \frac{\mu_0 F_x(\phi)}{g(\phi)} \qquad (4.4\text{-}20)$$

By (4.4-6), we have that the total field intensity may be expressed as the some of the individual fields, i.e.

$$H(\phi) = \frac{F(\phi)}{g(\phi)} \qquad (4.4\text{-}21)$$

and

$$B(\phi) = \frac{\mu_0 F(\phi)}{g(\phi)} \qquad (4.4\text{-}22)$$

where

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$$F(\phi) = \sum_{j=1}^{J} F_x(\phi) \qquad (4.4\text{-}23)$$

Thus, to compute the total airgap fields it is convenient to first find the MMF due to the individual windings; then find the total MMF using (4.4-23); and then find total field intensity and flux density using (4.4-21) and (4.4-22), respectfully.

4.5 ROTATING MMF

The overall goal of this chapter is to establish methods that can be used to determine electrical circuit models of electromechanical systems. To this end, we have just discussed how to calculate the MMF due to a distributed winding. In the next section, we will use this information in the calculation of inductances of distributed windings. However, before we proceed into that discussion, it will be interesting to pause for a moment, and consider our results thus far in regard to the operation of a machine.

4.5.1 Rotating MMF in Two-Phase Systems

Let us first consider a two-phase machine with two stator windings as and bs. Let us suppose that the turns density of these two windings may be expressed

$$n_{as} = N_s \sin(P\phi_{sm} / 2) \qquad (4.5\text{-}1)$$

$$n_{bs} = -N_s \cos(P\phi_{sm} / 2) \qquad (4.5\text{-}2)$$

From (4.3-15), it follows that

$$w_{as} = \frac{2N_s}{P} \cos(P\phi_{sm} / 2) \qquad (4.5\text{-}3)$$

$$w_{bs} = \frac{2N_s}{P} \sin(P\phi_{sm} / 2) \qquad (4.5\text{-}4)$$

where i_{as} and i_{bs} are the currents flowing into the two windings.

From (4.4-17) the MMF associated with these two windings may be expressed as

$$F_{as} = \frac{2N_s}{P} \cos(P\phi_{sm} / 2)\, i_{as} \qquad (4.5\text{-}5)$$

$$F_{bs} = \frac{2N_s}{P} \sin(P\phi_{sm} / 2)\, i_{bs} \qquad (4.5\text{-}6)$$

From (4.4-23), the total MMF associated with these two stator windings is given by

$$F = F_{as} + F_{bs} \qquad (4.5\text{-}7)$$

Combining (4.5-5)-(4.5-7) we have that

$$F = \frac{2N_s}{P}\left(\cos\left(P\phi_{sm}/2\right)i_{as} + \sin\left(P\phi_{sm}/2\right)i_{bs}\right) \tag{4.5-8}$$

As it turns out (4.5-8) is extremely useful in understanding the basic operation of ac electric machinery. Let us suppose that the a- and b-phase currents are controlled such that

$$i_{as} = \sqrt{2}I_s\cos(\omega_e t + \phi_i) \tag{4.5-9}$$

$$i_{bs} = \sqrt{2}I_s\sin(\omega_e t + \phi_i) \tag{4.5-10}$$

where I_s is the root-mean-square (rms) amplitude, ω_e is the radian frequency, and ϕ_i is an arbitrary phase. Substitution of (4.5-9) and (4.5-10) into (4.5-8) yields

$$F = \frac{2\sqrt{2}N_s I_s}{P}\cos\left(P\phi_{sm}/2 - \omega_e t - \phi_i\right) \tag{4.5-11}$$

The result (4.5-11) is an extremely important one. From (4.5-11), we can see that given a sinusoidal turns distribution (with the two windings in phase quadrature) and the two sinusoidal currents (again in phase quadrature) we arrive at a MMF which is sinusoidal in space (ϕ_{sm}) and sinusoidal in time (t). It represents a wave equation. In other words, the resulting MMF is a moving wave. To see this, consider the peak of the wave, wherein the argument to the cosine term in (4.5-11) is zero. In particular, at the peak of the wave

$$P\phi_{sm}/2 - \omega_e t - \phi_i = 0 \tag{4.5-12}$$

From (4.5-12) we have

$$\phi_{sm} = \frac{2}{P}\omega_e t + \frac{2}{P}\phi_i \tag{4.5-13}$$

Thus, the peak of the wave is moving at a speed of $2\omega_e/P$.

In synchronous machines, the rotor speed will be equal to the speed of the stator MMF wave, so it is clear that the speed will vary with ω_e. It can also be seen that as the number of poles is increased, the speed of the MMF wave (and hence the rotor) will decrease as well.

The creation of a MMF wave which travels in a single direction requires at least two currents, and two windings. To help see this, let us consider the situation when the b-phase current is set to zero; and everything else is kept the same. Substitution of (4.5-9) and $i_{bs} = 0$ into (4.5-8) yields

$$F=\frac{2\sqrt{2}N_s I_s}{P}\cos\left(P\phi_{sm}/2\right)\cos(\omega_e t+\phi_i) \qquad (4.5\text{-}14)$$

which may be expanded as

$$F=\frac{\sqrt{2}N_s I_s}{P}\cos\left(P\phi_{sm}/2-\omega_e t-\phi_i\right)+ \frac{\sqrt{2}N_s I_s}{P}\cos\left(P\phi_{sm}/2+\omega_e t+\phi_i\right) \qquad (4.5\text{-}15)$$

The first term represents a traveling wave moving at a speed of $2\omega_e/P$; the second term represents a traveling wave moving at a speed of $-2\omega_e/P$. The existence of this second traveling wave will normally significantly degrade the performance of the electromechanical device.

4.5.2 Rotating MMF in Three-Phase Systems

It is interesting to now consider a three phase winding. We will assume that the turns distribution for the stator windings of a certain three phase machine may be expressed.

$$n_{as}(\phi_{sm})=N_s\sin(P\phi_{sm}/2)+N_{s3}\sin(3P\phi_{sm}/2) \qquad (4.5\text{-}16)$$

$$n_{bs}(\phi_{sm})=N_s\sin(P\phi_{sm}/2-2\pi/3)+N_{s3}\sin(3P\phi_{sm}/2) \qquad (4.5\text{-}17)$$

$$n_{cs}(\phi_{sm})=N_s\sin(P\phi_{sm}/2+2\pi/3)+N_{s3}\sin(3P\phi_{sm}/2) \qquad (4.5\text{-}18)$$

In (4.5-16)-(4.5-18), the second term (the third harmonic term) is useful in achieving a more even slot fill.

In order to calculate the stator MMF for this system, we must first find the winding function. Using the methods of Section 4.4, the winding functions may be expressed

$$w_{as}(\phi_{sm})=\frac{2N_s}{P}\cos(P\phi_{sm}/2)+\frac{2N_{s3}}{3P}\cos(3P\phi_{sm}/2) \qquad (4.5\text{-}19)$$

$$w_{bs}(\phi_{sm})=\frac{2N_s}{P}\cos(P\phi_{sm}/2-2\pi/3)+\frac{2N_{s3}}{3P}\cos(3P\phi_{sm}/2) \qquad (4.5\text{-}20)$$

$$w_{cs}(\phi_{sm})=\frac{2N_s}{P}\cos(P\phi_{sm}/2+2\pi/3)+\frac{2N_{s3}}{3P}\cos(3P\phi_{sm}/2) \qquad (4.5\text{-}21)$$

The total stator MMF, will be the sum of the winding MMFs, thus

$$F=w_{as}(\phi_{sm})i_{as}+w_{bs}(\phi_{sm})i_{bs}+w_{cs}(\phi_{sm})i_{cs} \qquad (4.5\text{-}22)$$

To proceed further, we need to inject currents into the system. Let us consider a balanced three phase set of currents of the form

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$$i_{as} = \sqrt{2} I_s \cos(\omega_e t + \phi_i) \tag{4.5-23}$$

$$i_{bs} = \sqrt{2} I_s \cos(\omega_e t + \phi_i - 2\pi/3) \tag{4.5-24}$$

$$i_{cs} = \sqrt{2} I_s \cos(\omega_e t + \phi_i + 2\pi/3) \tag{4.5-25}$$

Substitution of (4.5-19)-(4.5-21) and (4.5-23)-(4.5-25) into (4.5-22) and simplifying yields

$$F = \frac{3\sqrt{2} N_s I_s}{P} \cos\left(P\phi_{sm}/2 - \omega_e t - \phi_i\right) \tag{4.5-26}$$

which, except for a factor of 3/2 in the amplitude, is the same as (4.5-11). Thus it can be seen that balanced 2-phase systems are very similar to balanced 3-phase systems, at least as far as the rotating MMF is concerned. Any deviation from a balanced three-phase excitation will result in forward and reverse components of the MMF wave.

4.6 FLUX LINKAGE AND MAGNETIZING INDUCTANCE

In the remaining sections of this chapter, we will start to fulfill the last of our goals for this chapter – that is the calculation of the electrical parameters of the machine. These parameters include the magnetizing inductances (self and mutual), leakage inductances (self and mutual), and resistances of the windings.

Before embarking on this process, it is appropriate to consider what is meant by magnetizing and leakage flux, and by magnetizing and leakage inductance. In essence, we can view the flux linking any winding in terms of two components

$$\lambda_{xy} = \lambda_{xym} + \lambda_{xyl} \tag{4.6-1}$$

where λ_{xym} is the magnetizing flux linkage and λ_{xyl} is the leakage flux linkage. The distinction between these two quantities is that the λ_{xym} is associated with radial flux flow across the airgap and links both the stator and rotor windings. Leakage flux is associated with flux that does not couple both the rotor and stator windings.

Associated with the concept of magnetizing and leakage flux are the concepts of magnetizing and leakage inductance. In particular, magnetizing inductance relates magnetizing flux to the current. In order to state these concepts mathematically, let α and β denote two windings (and will take on values of 'as', 'bs', 'cs', 'ar', 'br', and 'cr', etc). Then the magnetizing inductance between two windings may be expressed

$$L_{\alpha\beta m} = \frac{\lambda_{\alpha m}\big|_{\text{due to } i_\beta}}{i_\beta} \tag{4.6-2}$$

Similarly, the leakage inductance relates leakage flux to current.

$$L_{\alpha\beta l} = \frac{\lambda_{\alpha l}\big|_{due\ to\ i_\beta}}{i_\beta} \quad (4.6\text{-}3)$$

From our definition, the mutual leakage inductance between a stator winding and a rotor winding will be zero. However, there will be leakage inductance between different stator windings and between different rotor windings.

In this, we will attempt to calculate the magnetizing inductance between two windings. Our approach to the calculation of magnetizing inductance will be two fold. First, we will conduct an approximate analysis wherein we ignore the effects of slots and teeth. This will be useful in establishing the basic concepts. Next, we will modify our analysis to include these effects.

4.6.1 Initial Analysis

Our goal in this section is to find a method to calculate the inductance between two windings. To this end, consider Figure 4.6-1 which shows a stator. Consider the incremental area along the inner surface of the stator, as shown. The radius of this incremental section is r and its position is ϕ (Note that by this drawing ϕ is relative to the stator and could be designated ϕ_{sm}; however, as an identical argument could be made using any reference point, ϕ is used to denote position). The length of the edge segment is $rd\phi$. If the length of the stator is L, it follows that the incremental area is $Lrd\phi$. For small $d\phi$, the flux through this incremental area may be expressed

$$\Phi(\phi) = B(\phi)Lrd\phi \quad (4.6\text{-}4)$$

Now recall that the winding function $w_\alpha(\phi)$ describes the how many times a winding α links the flux at a position ϕ. Thus the contribution of the flux linking winding α through this incremental area may be expressed $w_\alpha(\phi)B(\phi)Lrd\phi$. Adding up all the incremental areas along the stator, we have

$$\lambda_{\alpha,m} = \int_0^{2\pi} B(\phi)w_\alpha(\phi)Lrd\phi \quad (4.6\text{-}5)$$

Now that we have a means to calculate the flux linkage given the flux density and winding function, recall from (4.4-20) that the flux density due to winding β may be expressed

$$B_\beta(\phi) = \frac{\mu_0}{g(\phi)} w_\beta(\phi)i_\beta \quad (4.6\text{-}6)$$

Substitution of (4.6-6) into (4.6-5) we have that the flux in winding α due to the current in winding β may be expressed

$$\lambda_{\alpha,m} = \int_0^{2\pi} \frac{\mu_0}{g(\phi)} w_\alpha(\phi) i_\beta w_\beta(\phi) rL d\phi \tag{4.6-7}$$

Rearranging (4.6-7) gives us a method for calculating the mutual inductance between two windings. In particular,

$$\frac{\lambda_{\alpha,m}}{i_\beta} = L_{m,\alpha\beta} = \mu_0 rL \int_0^{2\pi} \frac{w_\alpha(\phi) w_\beta(\phi)}{g(\phi)} d\phi \tag{4.6-8}$$

Let us now consider a brief example. Consider the a- and b- phase winding of a 2-pole 3-phase machine with a uniform airgap. Suppose that the turns density may be expressed

$$n_{as} = N_s \sin(P\phi_{sm}/2) \tag{4.6-9}$$

$$n_{bs} = N_s \sin(P\phi_{sm}/2 - 2\pi/3) \tag{4.6-10}$$

Using the techniques of Section 4.3, we have

$$w_{as} = \frac{2N_s}{P}\cos(P\phi_{sm}/2) \tag{4.6-11}$$

$$w_{bs} = \frac{2N_s}{P}\cos(P\phi_{sm}/2 - 2\pi/3) \tag{4.6-12}$$

Substitution of (4.6-11) and (4.6-12) into (4.6-8) and evaluating yields

$$L_{abs} = -\frac{2\pi\mu_0 rLN_s^2}{P^2 g} \tag{4.6-13}$$

4.6.2 Compensating for Slots – Carter’s Coefficient

In our previous analysis, we have neglected the effects of slots on the stator and rotor. As it turns out, the effects of slots can be readily incorporated into the analysis by replacing the airgap g with a modified airgap g'. In particular, for the case of the stator slots, the modified airgap is calculated as

$$g' = gc_s \tag{4.6-14}$$

where c_s is the stator Carter’s coefficient. We will now derive this result as well as a value for c_s.

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The derivation of (4.6-11) begins with consideration of Figure 4.6-2. This figure depicts the developed diagram over a small range of position $\Delta\phi$ corresponding to one half of a stator slot width plus one half of a stator tooth width. Thus

$$r\Delta\phi = \frac{1}{2}w_{ss} + \frac{1}{2}w_{st} \tag{4.6-15}$$

where w_{ss} is the stator slot width and w_{st} is the stator tooth width, both measured at the stator / airgap interface.

Let us first consider the situation if we ignore the slot. In this case, it can be shown that the flux flowing across the airgap in the interval $\Delta\phi$ may be expressed

$$\Phi = \frac{\mu_0 FL}{2g}(w_{ss} + w_{st}) \tag{4.6-16}$$

where L is the length of the machine. Because the slot is unaccounted for in (4.6-16), this expression is in error, because part of the flux (Φ_2) will have to travel further. Our goal will be to establish a value g' such that

$$\Phi = \frac{\mu_0 FL}{2g'}(w_{ss} + w_{st}) \tag{4.6-17}$$

is correct (or is at least a good approximation).

To this end, let us calculate the flux including the effects of the flux. To this end, it is convenient to divide the flux into two components,

$$\Phi = \Phi_1 + \Phi_2 \tag{4.6-18}$$

The first term is readily expressed

$$\Phi_1 = \frac{\mu_0 F w_{st} L}{2g} \tag{4.6-19}$$

The second term is more difficult. At a position z the distance from the rotor to the stator along the indicate path is $g + \pi z/2$. Thus, the field intensity along this path may be estimated as

$$H = \frac{F}{g + \pi z/2} \tag{4.6-20}$$

and the flux density as

$$B = \frac{\mu_0 F}{g + \pi z/2} \tag{4.6-21}$$

The flux Φ_2 may be expressed

$$\Phi_2 = \int_{z=0}^{w_{ss}/2} BLdz \tag{4.6-22}$$

Substitution of (4.6-21) into (4.6-22) yields

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$$\Phi_2 = \frac{2\mu_0 FL}{\pi}\ln\left(1+\frac{\pi\, w_{ss}}{4g}\right) \qquad (4.6\text{-}23)$$

The final step is to add (4.6-19) and (4.6-23) and to equate the result to (4.6-17). The result is (4.6-14) where

$$c_s = \frac{w_{ss}+w_{st}}{w_{st}+\frac{4g}{\pi}\ln\left(1+\frac{\pi w_{ss}}{4g}\right)} \qquad (4.6\text{-}24)$$

Observe that g, g', and c_s can all be functions of position (as measured from the stator or the rotor) but this functional dependence is not explicitly shown.

The use of (4.6-14)/(4.6-24) is straightforward and very useful, because it allows us, with a simple substitution of g' for g, to account, albeit approximately, for the effects of the stator slots.

For machines with both stator and rotor slots, the concept of Carter's coefficient can still be used; however, in this case

$$g' = gc_s c_r \qquad (4.6\text{-}25)$$

where

$$c_r = \frac{w_{rs}+w_{rt}}{w_{rt}+\frac{4gc_s}{\pi}\ln\left(1+\frac{\pi w_{rs}}{4g}\right)} \qquad (4.6\text{-}26)$$

and where w_{rs} and w_{rt} are the width of the rotor slot and rotor tooth where it meets the airgap.

4.7 LEAKAGE INDUCTANCE

In this section, we calculate the leakage inductance of a winding. The leakage inductance relates the flux produced by a stator (rotor) winding that does not link the rotor (stator). Although we normally think of a leakage flux as flux that does not couple any other winding, and which can be described in terms of a self leakage inductance. However, this is not the case with polyphase windings. For example, in a three-phase machine, current in the a-phase of the stator will produce flux in the b-phase of the stator, but does not couple to any of the rotor windings. This flux may be described in terms of a mutual leakage inductance. In this section, we will find expressions for both the self and mutual leakage inductance.

4.7.1 Energy Stored in a Slot

We begin our discussion of leakage inductance by considering the inductance associated with N conductors in a single slot, as depicted in Figure 4.7-1. Therein, w_s and d_s denote the width and depth of the slot, and d_w is the depth of the winding within the slot. The dominant path for leakage flux is indicated. It may

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surprise the reader that the leakage path makes use of the surrounding iron – but this is in fact the path of least resistance (or reluctance, in this case), and so it is the dominant path of the leakage flux. Ignoring the reluctance of the iron, we can readily establish that the field intensity in the slot may be expressed

$$H = \begin{cases} \dfrac{di_{slt}}{d_w w_s} & 0 \le d \le d_w \\ \dfrac{i_{slt}}{w_s} & d_w \le d \le d_s \end{cases} \tag{4.7-1}$$

where i_{slt} is the current in the slot.

Our next step will be to determine the energy stored by the leakage field. In general, the energy stored by a magnetic field may be expressed

$$E = \frac{1}{2} \int_{volume} \mathbf{B} \cdot \mathbf{H} dv \tag{4.7-2}$$

In our case B and H can be viewed as scalars; since $B = \mu_0 H$ in the slot, and taking the incremental volume to be the length L times the slot width w_s times an incremental depth d we have

$$E = \frac{1}{2} \int_0^{d_s} \mu_0 H^2 L w_s dd \tag{4.7-3}$$

Substitution of (4.7-1) into (4.7-3) and evaluating yields

$$E = \frac{\mu_0 L}{2 w_s} \left(d_s - \frac{2}{3} d_w \right) i_{slt}^2 \tag{4.7-4}$$

4.7.2 Leakage Inductances Using Discrete Winding Description

The slot current is composed of currents from all the phases within that slot. Let us consider the situation in which two windings are present, denoted winding α and winding β. The slot current in the i'th may be expressed

$$i_{slt} = N_{\alpha,i} i_\alpha + N_{\beta,i} i_\beta \tag{4.7-5}$$

Substitution of (4.7-5) into (4.7-4) yields

$$E = \frac{\mu_0 L}{2w_s}\left(d_s - \frac{2}{3}d_w\right)\left(N_{\alpha,i}^2 i_\alpha^2 + 2N_{\alpha,i} i_\alpha N_{\beta,i} i_\beta + N_{\beta,i}^2 i_\beta^2\right) \tag{4.7-6}$$

From our work in Section 4, the energy expressed in the slot (which may be viewed as a linear leakage inductance) may be expressed

$$E = \frac{1}{2}\begin{bmatrix} i_\alpha & i_\beta \end{bmatrix}\begin{bmatrix} L_{l\alpha\alpha,i} & L_{l\alpha\beta,i} \\ L_{l\alpha\beta,i} & L_{l\beta\beta,i} \end{bmatrix}\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \tag{4.7-7}$$

and expanded as

$$E = \frac{1}{2}\left(L_{l\alpha\alpha,i} i_x^2 + 2L_{l\alpha\beta,i} i_\alpha i_\beta + L_{l\beta\beta,i} i_\beta^2\right) \tag{4.7-8}$$

Comparing (4.7-6) to (4.7-8) we obtain the self leakage inductance of the i'th slot of winding x and the mutual leakage inductance between winding x and winding y. In particular,

$$L_{l\alpha,i} = \frac{\mu_0 L}{w_s}\left(d_s - \frac{2}{3}d_w\right)N_{\alpha,i}^2 \tag{4.7-9}$$

$$L_{l\alpha\beta,i} = \frac{\mu_0 L}{w_s}\left(d_s - \frac{2}{3}d_w\right)N_{\alpha,i} N_{\beta,i} \tag{4.7-10}$$

Using (4.7-9) and (4.7-10), we can calculate the self leakage inductance of a slot of any winding, as well as the mutual leakage inductance between any two windings.

In order to calculate the total leakage inductance of the winding, it is sufficient to add the leakage inductance over every slot. Thus we have that

$$L_{l\alpha\alpha} = \sum_{j=1}^{N_{slt}} L_{l\alpha\alpha,j} \tag{4.7-11}$$

$$L_{l\alpha\beta} = \sum_{j=1}^{N_{slt}} L_{l\alpha\beta,j} \tag{4.7-12}$$

4.7.3 Leakage Inductance Using Continuous Winding Description

As it turns out (4.7-9)-(4.7-12) can be readily modified for use with a continuous winding description. To this end, let us artificially divide the machine into N_{slt} slots. Define $\Delta\phi$ to be the angle spanned by one tooth and one slot. Assuming $\Delta\phi$ is small, we have

$$r\Delta\phi = w_s + w_t \tag{4.7-13}$$

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which may be rearranged as

$$w_s = \frac{r\Delta\phi}{(1 + w_t / w_s)} \tag{4.7-14}$$

This rearrangement may seem strange at the present however, we will be performing limiting operations wherein $\Delta\phi$ goes to zero, wherein w_s will also go to zero. However, the ratio of w_s / w_t will be a constant.

Next, we can approximate the number of turns in our *i*'th artificial slot as

$$N_{\alpha,i} = n_\alpha(\phi_{sm,i})\Delta\phi \tag{4.7-15}$$

where $\phi_{sm,i}$ is defined by (4.2-2).

Finally, note that

$$N_{slt} = \frac{2\pi}{\Delta\phi} \tag{4.7-16}$$

Our next step is the substitution of (4.7-9) into (4.7-11) which yields

$$L_{l\alpha\alpha} = \mu_0 L\left(d_s - \frac{2}{3}d_w\right)\sum_{j=1}^{N_{slt}} \frac{N_{\alpha,i}^2}{w_s} \tag{4.7-17}$$

Substitution of (4.7-14)-(4.7-16) into (4.7-17) yields

$$L_{l\alpha\alpha} = \mu_0 L\left(d_s - \frac{2}{3}d_w\right)\left(1 + \frac{w_t}{w_s}\right)\sum_{j=1}^{N_{slt}} \frac{n_{\alpha,i}^2(\phi_{sm,i})\Delta\phi}{r} \tag{4.7-18}$$

Finally, in the limit as $\Delta\phi \to 0$ (and $N_{slt} \to \infty$) we have

$$L_{l\alpha\alpha} = \frac{\mu_0 L}{r}\left(d_s - \frac{2}{3}d_w\right)\left(1 + \frac{w_t}{w_s}\right)\int_0^{2\pi} n_\alpha^2(\phi)d\phi \tag{4.7-19}$$

Utilizing the same approach, we can also show that

$$L_{l\alpha\beta} = \frac{\mu_0 L}{r}\left(d_s - \frac{2}{3}d_w\right)\left(1 + \frac{w_t}{w_s}\right)\int_0^{2\pi} n_\alpha(\phi)n_\beta(\phi)d\phi \tag{4.7-20}$$

4.8 RESISTANCE

At this section, we consider the problem of finding the resistance of a winding. Our approach here will focus on the discrete winding description. A continuous winding description can readily be converted to this form using the methods of Section 4.2.4. To calculate the resistance of a winding, we will first find the total volume of conductor utilized by the winding. Assuming we know the cross-sectional area of the conductor, we can then readily establish the length and resistance.

The volume of a winding can be broken down into two parts – the volume located in the slots, and the volume located in the end-turns. The slot volume of winding xy is equal to the sum of the number of conductors in each slot times the volume of each conductor (which is, in turn, the length times the cross sectional area). Thus,

$$v_{slot} = (L + 2L_e)a_c \sum_{i=1}^{N_{yslt}} \left| N_{xy,i} \right| \quad (4.8\text{-}1)$$

where L_e represents the distance between the end of the iron of the machine and the point where the end turns begin. This gap is needed for the turning radius of the conductor and as a secondary result of automated winding machinery used in the manufacturing process.

Similarly, the volume of copper associated with the end turn region may be expressed

$$v_{end} = r_{ymn} \frac{2\pi}{N_{yslt}} a_c \sum_{i=1}^{N_{yslt}} \left| M_{xy,i} \right| \quad (4.8\text{-}2)$$

In (4.8-2), r_{ymn} is the mean radius (from the center of the machine) of an end turn; this radius times the angle between slots yields the length between slot centers; this length multiplied by the cross sectional area of the conductor yields an estimate of the volume of each conductor in the segment; this is multiplied by the number of turns and summed over all segments to yield the total end turn conductor volume.

Since there are two end turn regions, the total conductor volume associated with the a-phase may be expressed as

$$v_{total} = v_{slot} + 2v_{end} \quad (4.8\text{-}3)$$

Since we know the conductor cross section, the length of the conductor associated with the winding may be expressed as

$$l_c = \frac{v_{total}}{a_c} \quad (4.8\text{-}4)$$

whereupon the phase resistance may be calculated as

$$R = \frac{l_c}{a_c \sigma_c} \tag{4.8-5}$$

where σ_c is the conductivity of the conductor.

4.9 ACKNOWLEDGEMENTS

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4.10 REFERENCES

[1] D.C. Hanselman, *Brushless Permanent-Magnet Motor Design*, McGraw-Hill, 1994.

AUTHORS NOTES (TO HIMSELF)

In the future, you may want to add problems, as well as a section on distributed circuit (skin and eddy) effects. Also, it would be nice to add resistance of continuous windings. Another thought – add permeance formula for trapezoidal slot. Also, just say it is okay to use both forms of the winding description, and then go back and forth in terms of what is easiest.

Explain that there is some choice made in the assumed symmetry of the winding function.

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