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Deterministic chaos in nonlinear optical wave-mixing

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Abstract. Propagation in a Kerr medium of a system of equal-frequency light waves has been studied, each of the waves being involved in two four-wave mixing processes. Competition of these processes is shown to result in non-integrability and chaos. A stochastic regime of evolution sets in if the input values of the wave amplitudes differ strongly enough from the ones corresponding to the integrable limit connected with the system symmetry. Another condition for the chaos—strong energy exchange between the waves—is realized when the medium and radiation parameter values have the same order of magnitude as those required to obtain enhanced phase-conjugate reflection due to degenerate four-wave mixing. Various manifestations of stochastic instability of the system of light waves under experimental conditions are also discussed.

1. Introduction

Recently a new research area has emerged from nonlinear optics and the theory of dynamic systems, namely the search for and investigation of deterministic chaos [1] in nonlinear optical processes (optical chaos, or optical turbulence) [2]. This rapidly developing branch of quantum electronics has already involved such studies in laser generation, light–matter interaction under optical multistability, coherent co-operative nonlinear optical phenomena, interaction of laser radiation with complex molecules [2], and so on.

An area that has been comparatively less well investigated is the chaotic behaviour of systems of interacting light waves [3–7]. Under conditions when dissipative processes are essential this problem has been considered for optical second-harmonic generation inside an external cavity [3] and for stimulated Raman light scattering in the active medium [4]. Chaotic emission of a strange attractor type was experimentally observed [5] under phase conjugation of laser radiation in a medium with a non-instantaneous nonlinear response. Of particular interest are the conditions leading to the appearance of optical chaos in systems of interacting light waves in the Hamiltonian regime [6, 7]. In [6] the possibility of chaotic polarization behaviour is shown for counterpropagating light waves in an anisotropic medium. Transition to chaos arising from interference of plane waves in a nonlinear medium is discussed in [7].

This paper studies the conditions for the appearance of stochastic instability in a system of six light waves parametrically interacting in a transparent medium via a third-order nonlinearity. Such a system is one of the simplest examples of an ensemble of interacting light waves allowing Hamiltonian chaos. Besides, situations similar to the one discussed take place in experiments on effective self-diffraction and optical phase conjugation (OPC) of laser radiation [8].

Section 2 of this paper formulates the problem of dynamics of six light waves, each of them being involved in two degenerate four-wave mixing processes. The initial system of reduced wave equations has been transformed into the Hamiltonian one with two degrees of freedom typical of nonlinear dynamics and in the general case non-integrable. In section 3 the range of boundary values of the wave amplitudes where stochastic instability arises has been found from the conditions of violation of the integrable limit. The concluding part of the paper discusses the peculiarities of experimental manifestation of chaos.

2. Basic equations

Consider spatial evolution of a system of six plane light waves propagating in a transparent medium with instantaneous Kerr nonlinearity. Suppose this system possesses the following properties: the wave frequencies are identical and equal to ω; the wave-vectors $\mathbf{k}_j$ $(j=1,2,\ldots,6)$ belong to the surface of a cone with the apex angle α (see figure 1); the angles between the wave-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$; $\mathbf{k}_3$ and $\mathbf{k}_4$; $\mathbf{k}_5$ and $\mathbf{k}_6$ are also equal to α; angle α is sufficiently small to consider the waves as co-polarized $(\alpha \ll \pi)$. Projections of the total electric field strength and of the nonlinear medium polarization on the polarization direction of the waves are represented as

$$E=\tfrac{1}{2}A\exp(\mathrm{i}\omega t)+\mathrm{c.c.},\qquad A=\sum_{j=1}^{6}E_j\exp(-\mathrm{i}\mathbf{k}_j\cdot\mathbf{r}), \tag{1}$$

$$P=\tfrac{1}{2}p\exp(\mathrm{i}\omega t)+\mathrm{c.c.},\qquad p=\varkappa^{(3)}|A|^2A,\qquad \mathrm{Im}\,\varkappa^{(3)}=0. \tag{2}$$

Using (1) and (2) in a conventional way [9], we obtain a system of reduced wave equations:

$$\left.\begin{aligned}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}+\tfrac{1}{2}|\varepsilon_{1,2}|^2\right)\varepsilon_{1,2}&=(\varepsilon_3\varepsilon_4+\varepsilon_5\varepsilon_6)\varepsilon^*_{2,1};\\ \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}+\tfrac{1}{2}|\varepsilon_{3,4}|^2\right)\varepsilon_{3,4}&=(\varepsilon_1\varepsilon_2+\varepsilon_5\varepsilon_6)\varepsilon^*_{4,3};\\ \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}+\tfrac{1}{2}|\varepsilon_{5,6}|^2\right)\varepsilon_{5,6}&=(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4)\varepsilon^*_{6,5};\end{aligned}\right\} \tag{3}$$

$$\xi=\frac{z}{l},\qquad l=\frac{cn}{4\pi\omega\varkappa^{(3)}|E_{10}|^2},\qquad E_{10}=E_1(z=0),$$

$$\varepsilon_j=\frac{E_j}{E_{10}}\exp(\mathrm{i}L\xi), \tag{4}$$

$$L=\sum_{j=1}^{6}\frac{|E_j|^2}{|E_{10}|^2}=\sum_{j=1}^{6}|\varepsilon_j|^2. \tag{5}$$

Here the length l is the spatial scale of the energy exchange between the waves; c is the light velocity in a vacuum; n is the linear refractive index; E_{10} is the complex amplitude of one of the waves at the medium boundary; L is the normalized total intensity of the waves. Equations (3) have been obtained under exact phase-matching conditions for the degenerate four-photon parametric processes

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4,\qquad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_5+\mathbf{k}_6,\qquad \mathbf{k}_3+\mathbf{k}_4=\mathbf{k}_5+\mathbf{k}_6.$$

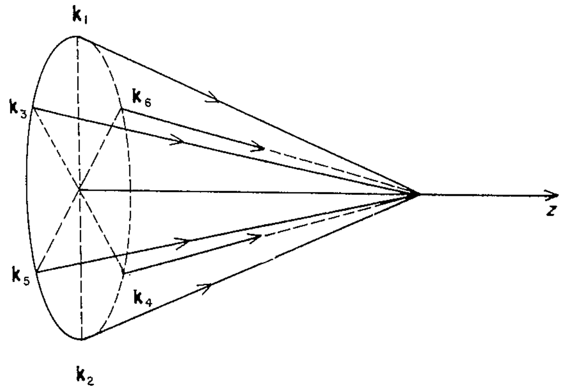

Figure 1. Wave-vectors of six plane light waves propagating in the nonlinear medium.

The magnitude of α is assumed to have the lower limit $\alpha \gg \pi(|\mathbf{k}_j|l)^{-1/2}$ which prevents generation of the new modes of optical field due to stimulated parametric scattering of the waves taken into account in (1). Each of the six waves is simultaneously involved in two four-wave mixing processes.

The most interesting feature of system (3) is the occurrence of multiple four-wave interactions. In this regard the system models the situation arising from self-diffraction of laser radiation [8] (but is simpler). The number of waves involved in (3) is fixed initially whereas in the case of self-diffraction a variety of waves is generated, corresponding to diffraction maxima of different orders. Notice that the system of six interacting waves (3) can be realized by injecting into a nonlinear medium four light beams with, for instance, the wave-vectors $\mathbf{k}_{1,2,3,5}$ (the waves corresponding to $\mathbf{k}_{4,6}$ will arise in the processes $\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_{3,5}\rightarrow\mathbf{k}_{4,6}$). Thus the required optical scheme is close to that used to study four-wave mixing under non-complanar phase-matching conditions, when three light beams are injected into a nonlinear medium [10].

Equations (3) together with the complex-conjugate ones may be treated as Hamiltonian equations

$$\mathrm{i}\dot{\varepsilon}_j=\frac{\partial\mathscr{H}}{\partial\varepsilon_j^*},\qquad \mathrm{i}\dot{\varepsilon}_j^*=-\frac{\partial\mathscr{H}}{\partial\varepsilon_j},\qquad \dot{\cdot}\equiv\frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{6}$$

with the Hamiltonian

$$\mathscr{H}-\sum_{j=1}^{6}\frac{|\varepsilon_j|^4}{4}+(\varepsilon_1\varepsilon_2\varepsilon_3^*\varepsilon_4^*+\varepsilon_1\varepsilon_2\varepsilon_5^*\varepsilon_6^*+\varepsilon_3\varepsilon_4\varepsilon_5^*\varepsilon_6^*+\text{c.c.}). \tag{7}$$

System (6) has three motion integrals of the Manley–Rowe type

$$R_{1,2,3}=|\varepsilon_{1,3,5}|^2-|\varepsilon_{2,4,6}|^2 \tag{8}$$

and also L and $\mathscr{H}$.

In the appendix it is shown that system (3) can be reduced to the effective Hamiltonian system with two degrees of freedom with the new canonical variables J_n, ψ_n $(n=1,2)$:

$$\dot{J}_n=-\frac{\partial\mathscr{H}_{\mathrm{R}}}{\partial\psi_n},\qquad \dot{\psi}_n=\frac{\partial\mathscr{H}_{\mathrm{R}}}{\partial J_n}, \tag{9}$$

$$\mathscr{H}_{\mathrm{R}}=f_0(J_1,J_2)+f_1(J_1,J_2)\cos\psi_1+f_2(J_1,J_2)\cos\psi_2+f_3(J_1,J_2)\cos(\psi_2-\psi_1). \quad (10)$$

Therein one can also find the explicit form of the functions $f_i(J_1, J_2)$ $(i=0, 1, 2, 3)$ and the connection between the variables ε_j and J_n, ψ_n.

Hamiltonian (10) with an arbitrary dependence of f_i on J_n is quite typical of the problems of nonlinear dynamics [11]. It is known that for some types of functions f_i (J_1, J_2) the system with Hamiltonian (10) may exhibit chaotic behaviour [12, 13]. To date there are no general analytical techniques which could yield the criterion for transition to chaos in a system (9), (10). The Kolmogorov–Arnold–Mozer (KAM) theorem is not adequate in this case, and the absence of a small parameter makes it impossible to apply the perturbation technique. For this reason the dynamics of the system discussed was analysed numerically. To find the conditions for transition to chaos, the initial system of wave equations (3) was analysed, since it is more convenient for numerical analysis than (9), (10).

The numerical results obtained will be presented in the next section of the paper. Here we describe some cases when the dynamics of system (3) may be expected to be regular. These are the cases when the boundary values of the amplitudes (initial conditions) coincide for two pairs of the waves. We mean coincidence of the type

$$\varepsilon_k(0)=\varepsilon_m(0), \qquad \varepsilon_l(0)=\varepsilon_g(0), \quad (11)$$

where $(k, l), (m, g)=(1, 2), (3, 4), (5, 6)$, for example $\varepsilon_1(0)=\varepsilon_3(0), \varepsilon_2(0)=\varepsilon_4(0)$. Under these conditions the symmetry of system (3) allows us to put $\varepsilon_k(\xi)=\varepsilon_m(\xi)$; $\varepsilon_l(\xi)=\varepsilon_g(\xi)$ and thereby to reduce it to the equations

$$\left.\begin{aligned}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}+\tfrac{1}{2}|\varepsilon_{\mu,\nu}|^2\right)\varepsilon_{\mu,\nu}&=2\varepsilon_k\varepsilon_l\varepsilon^*_{\nu,\mu};\\ \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}+\tfrac{1}{2}|\varepsilon_{k,l}|^2-|\varepsilon_{l,k}|^2\right)\varepsilon_{k,l}&=\varepsilon_\mu\varepsilon_\nu\varepsilon^*_{l,k}.\end{aligned}\right\} \quad (12)$$

Indices μ, ν refer to the pair of waves on which no limitations have been imposed. System (12) has four motion integrals: L, $\mathscr{H}$ (in which now $\varepsilon_k=\varepsilon_m, \varepsilon_l=\varepsilon_g$) and two integrals (8) (the third one is identical to the first of those two) and hence is completely integrable in an eight-dimensional phase space. The numerical results show that the regular dynamics of system (12) correspond to a stable region of twelve-dimensional phase space of the initial system (3). Conditions for destroying the integrable limit (12) may be used to determine numerically the range of $\varepsilon_j(0)$ where system (3) has chaotic dynamics.

3. Numerical analysis

To find and study deterministic chaos, the set of equations (3) was solved numerically. The accuracy of calculation was controlled by checking whether the motion integrals (5), (7) and (8) were conserved. In all the cases their deviations from the values calculated at the medium boundary did not exceed 0·1%. To identify the chaotic realizations, apart from the solutions $\varepsilon_j(\xi)$ of system (3), the value

$$U(\xi)=\ln\Delta(\xi), \qquad \Delta(\xi)=\left[\sum_{j=1}^{6}|\varepsilon_j(\xi)-\varepsilon'_j(\xi)|^2\right]^{1/2} \quad (13)$$

was also calculated, where $\varepsilon'_j(\xi)$ were the solutions of (3) with the initial conditions close to $\varepsilon_j(0)$.

One of the main indications of chaos is the local instability of the dynamic system solutions—exponential divergence of close phase trajectories at distances comparable to the typical dimension of the phase volume of the system [1]. In the case under discussion this corresponds to the linear (on average) growth of the $U(\xi)$ value up to $U(\xi)\sim 1$.

Some typical $|\varepsilon_1(\xi)|^2$ and $U(\xi)$ dependences are shown in figures 2–6. Figure 2 illustrates the chaotic spatial dynamics of the normalized intensity of one of the interacting waves $|\varepsilon_1(\xi)|^2 = I_1(\xi)/I_1(0)$ ($I_1 = (c/8\pi)|E_{10}|^2$) is the wave intensity) and figure 3 shows local instability of the respective solutions of (3). A characteristic example of quasi-periodic dynamics is represented in figure 4. In this case the local instability is absent (figure 5).

The table summarizes the conditions of appearance of chaos which have been found from the numerical analysis. It shows that the waves dynamics are chaotic in a wide range of boundary values of their amplitudes, including the cases of absence of one (case 1 of the table) or two (case 2) waves at the medium boundary. From the table one can see that the dynamics of the light waves are regular in the integrable regimes described by (12) and close to it in the initial conditions. When the initial conditions are chosen in the regions sufficiently distant from the ones satisfying (11), the system exhibits a spatially chaotic behaviour. For instance, case 1 in the table indicates the absence of chaos at $\varepsilon_4(0)<0{\cdot}4$ (the initial conditions are close to $\varepsilon_3(0)=\varepsilon_5(0)=1$; $\varepsilon_4(0)=\varepsilon_6(0)=0$) and $\varepsilon_4(0)>0{\cdot}85$ (the conditions close to $\varepsilon_1(0)=\varepsilon_3(0)=1$; $\varepsilon_2(0)=\varepsilon_4(0)=1$) and its presence within the range $0{\cdot}4<\varepsilon_4(0)<0{\cdot}85$. Notice that coincidence of the initial conditions inside the pairs of waves 1 and 2, 3 and 4, 5 and 6 does not reduce system (3) to the integrable one. This follows, in particular, from the existence of chaos under $\varepsilon_1(0)=\varepsilon_2(0)=1$; $\varepsilon_3(0)=\varepsilon_4(0)=0{\cdot}5$ (case 3 in the table).

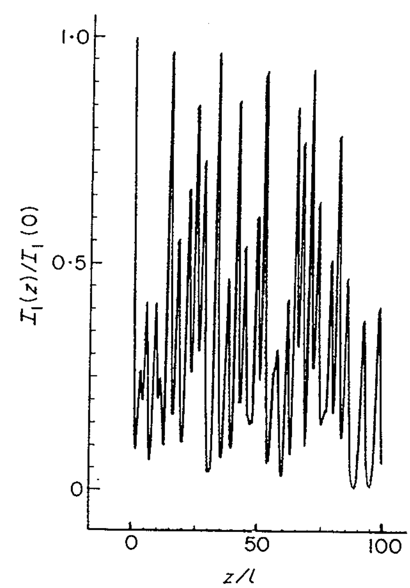

Figure 2. Spatial dependence of the intensity of one of the interacting waves $I_1(z)$ under conditions of chaos: $\varepsilon_{1,2}(0)=1$; $\varepsilon_3(0)=0{\cdot}9$; $\varepsilon_{4,6}(0)=0$; $\varepsilon_5(0)=0{\cdot}1$.

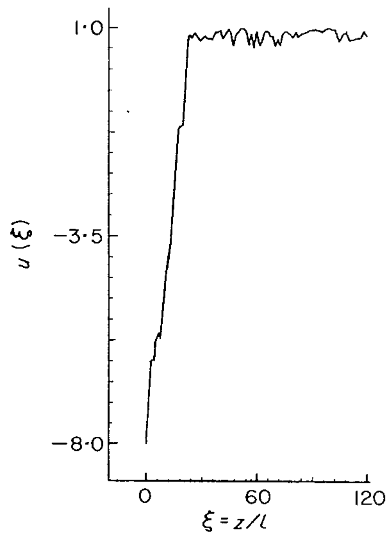

Figure 3. Linear (on average) growth of the $U(\xi)$ function up to unity—local instability of system (3)—in the conditions of figure 2.

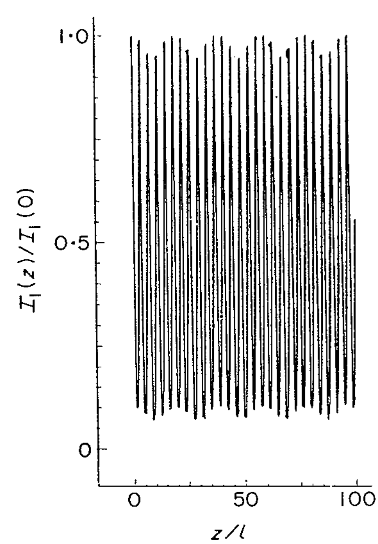

Figure 4. Regular quasi-periodic dependence $I_1(z)$ at $\varepsilon_{1,2}(0)=1$; $\varepsilon_{3,4,5}(0)=0{\cdot}5$; $\varepsilon_6(0)=0{\cdot}1$.

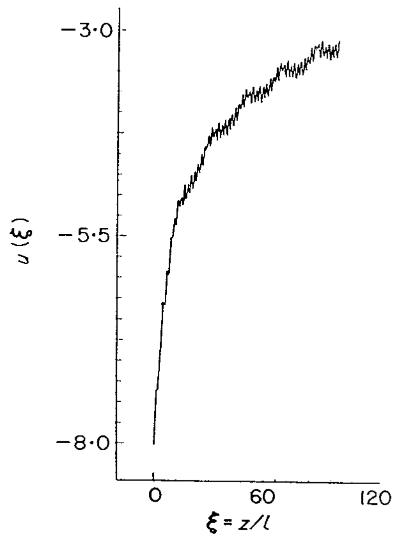

Figure 5. Absence of local instability of system (3) at boundary values of the waves amplitudes $\varepsilon_j(0)$, corresponding to figure 4.

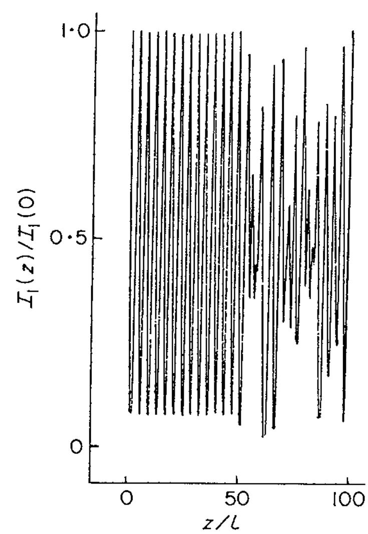

Figure 6. Realization of $I_1(z)$ at the initial conditions $\varepsilon_{1,2}(0)=1$; $\varepsilon_{3,4,5}(0)=0{\cdot}5$; $\varepsilon_6(0)=0{\cdot}2$, corresponding to the boundary between the regions of regular and chaotic spatial evolution.

The boundaries of ranges of chaos represented in the table are not strictly determined. The boundary between the regions of regular and chaotic motion in a phase space has a rather complicated structure and is characterized by an alternative sequence of regular and chaotic regions of small dimensions. Moreover, inside the chaotic regions there may exist small regions of regular motion—islands of stability [1]. Figure 6 illustrates a realization of the intensity of one of the waves under the initial conditions corresponding to the boundary between regular and chaotic motion. First, the wave intensity varies quasi-periodically, then it exhibits a number of random jumps. This leads us to suppose that transition to chaos in the system discussed proceeds via intermittency [1].

4. Discussion

The main results obtained may be summarized as follows. A steady-state spatial evolution of an ensemble of equal-frequency plane light waves has been studied in a transparent medium with an instantaneous Kerr nonlinearity. Each of the waves is supposed to be involved in two four-wave mixing processes simultaneously. Adequate physical conditions are realized, for example, in self-diffraction and phase conjugation of laser radiation at small angles between the interacting light beams. Spatial evolution of such an ensemble of waves is shown to be chaotic over a wide range of the boundary values of their amplitudes. In particular, the waves evolve chaotically if the amplitudes at the entrance to the medium occur in the region well distanced from the values corresponding to the integrable limit, whose existence is connected with the system symmetry. The physical reason for the appearance of chaos is the competition of parametric processes under strong energy exchange between the waves.

Regions of boundary values of the waves amplitudes $\varepsilon_j(0)$ corresponding to chaotic and regular spatial evolution.

	Studied region				
Case	$\varepsilon_3(0)$	$\varepsilon_4(0)$	$\varepsilon_5(0)$	$\varepsilon_6(0)$	Region of chaos
1	1	0–1	1	0	$\varepsilon_4(0)=0{\cdot}4$–$0{\cdot}85$
2	0·1–0·9	0	$1-\varepsilon_3(0)$ $=0{\cdot}9$–$0{\cdot}1$	0	$\left\{\begin{array}{l}\varepsilon_3(0)=0{\cdot}1\text{–}0{\cdot}4\\ \varepsilon_5(0)=0{\cdot}9\text{–}0{\cdot}6\end{array}\right.$ $\left\{\begin{array}{l}\varepsilon_3(0)=0{\cdot}6\text{–}0{\cdot}9\\ \varepsilon_5(0)=0{\cdot}4\text{–}0{\cdot}1\end{array}\right.$
3	0·5	0·5	0·5	0·1–1	$\varepsilon_6(0)=0{\cdot}2$–$0{\cdot}48$

In the table zero initial phases have been chosen for all the waves; $\varepsilon_{1,2}(0)=1$. The data summarized in the table indicate that spatial chaos sets for a wide range of boundary values of the interacting waves amplitudes. Case 2, for example, demonstrates chaos under initial conditions rather different from the values $\varepsilon_3(0)=\varepsilon_5(0)=0{\cdot}5$; $\varepsilon_4(0)=\varepsilon_6(0)=1$, satisfying requirements (11) of the integrable limit (12).

To conclude our discussion we give some remarks concerning the manifestation of stochastic instability of light waves in experiments. Spatial dynamics of system (3) in a chaotic regime are extremely sensitive to the choice of the initial conditions, which manifests itself in the local instability of the solutions of (3). Propagation of waves at distances of several characteristic lengths l is accompanied by an order of magnitude increase in the uncertainties of their amplitudes. Under experimental conditions this would lead to increased fluctuations in the system and a broadened radiation spectrum.

Another aspect of manifestation of stochastic instability is the following. Interaction of not too short light pulses can be studied in a quasi-steady-state approximation. In the case of interest this means the validity of formulating the problem in the form of (3) with the time-dependent boundary conditions: $\varepsilon_j(0)=\varepsilon_j(0,t)$. Then the normalizing constant E_{10} in the expression for the dimensionless dynamic variable ξ (see (4)) will have different values for various points of the pulse time envelopes: $E_{10}=E_{10}(t)$. For such points, at a fixed geometrical length of the medium l_m, the normalized length $\xi_m=l_m/l$ appears to be different too. It varies from zero (in the far wings of the envelopes) to maximum values corresponding to the maximum of radiation intensity. This leads us to suggest that the time envelopes of the interacting light beams would reproduce the specific features of the steady-state spatial evolution discussed above, and in particular the chaotic behaviour. This problem will be treated in detail elsewhere.

From figures 2 and 4 one can see that in order to identify the two types of evolution of waves, namely chaotic and quasi-periodic, one should have realizations of length

$$\xi_m \gtrsim 10. \tag{14}$$

Condition (14) is fulfilled at the values of medium and radiation parameters close to those required for enhanced phase-conjugate reflection by degenerate four-wave mixing, which is realized at $\xi_m>\pi/4$ [14, 15].

Notice that the Kerr nonlinearity responsible for parametric wave interaction leads as well to the effects of self-focusing/defocusing, neglected in (3). In self-focused beams the maximal increment g of growth of amplitudes of transverse field distribution perturbations is of order l^{-1} [16]. At medium lengths, $\xi_m\gtrsim 10$, the transverse field profile appears to be severely distorted, which manifests itself as small-scale self-focusing. For these conditions the initial model (3) seems ill-suited.

The situation changes dramatically for a self-defocusing effect. This process smoothes out inhomogeneities in a transverse optical field distribution and is self-limiting. Self-defocusing accounts for the increase in the divergence angles of initially collimated optical beams up to maximum values amounting (at $l_m\to\infty$) to $\theta_{NL}\sim(8\pi\chi^{(3)})^{1/2}E$. For $\alpha>\theta_{NL}$ model (3) proves to successfully describe main features of the dynamics of interacting waves. Estimates show that the conditions $\alpha>\theta_{NL}$ and $\xi_m\sim 10$ can be provided by using standard dye-lasers and sodium vapour as a nonlinear medium.

Within the framework of the optical scheme under study (see figure 1), a regime of parametric amplification is realized for any pair of weak waves of frequency ω, the wave-vectors $\mathbf{k}, \mathbf{k}'$ of which belong to the cone surface and are directed so that the angle between $\mathbf{k}$ and $\mathbf{k}'$ equals α. Numerical analysis shows that intensities of such waves grow exponentially with the increment $G\sim 2l^{-1}$. For Raman light scattering, where the intensity of the weak Stokes component grows exponentially too, the criterion of its effective generation is $Gl_m \sim 20$–30 [17]. Taking these values of the 'enhancement coefficient' Gl_m as the criterion for the case under discussion, one may expect that at medium lengths $l_m \sim 10l$ ($Gl_m \sim 20$), required to identify chaos, the field energy would remain accumulated in the modes considered in model (3).

The results obtained may be generalized by suggesting that observation of Hamiltonian (non-dissipative) optical chaos is possible not only in the six-beam interaction under consideration but also in the general case of competition of optical wave-mixing processes, such as under effective nonlinear conversion of multi-frequency laser radiation.

Acknowledgements

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Appendix

Systems (9) and (10) are derived as follows. Introduce new canonical variables ψ_j and J_j related to ε_j by

$$\left.\begin{aligned}
&\psi_1=\theta_1+\theta_2-\theta_3-\theta_4, && J_1=I_1+I_5,\\
&\psi_2=\theta_1+\theta_2-\theta_5-\theta_6, && J_2=-I_5,\\
&\psi_3=\theta_3, \quad \psi_4=\theta_2, && J_3=I_1+I_3+I_5, \quad J_4=I_2-I_1;\\
&\psi_5=\theta_4, \quad \psi_6=\theta_6, && J_5=I_1+I_4+I_5, \quad J_6=I_6-I_5;\\
&\theta_j=-\arg\varepsilon_j, && I_j=\tfrac{1}{2}|\varepsilon_j|^2.
\end{aligned}\right\} \tag{A 1}$$

Then system (6), (7) (equivalent to (3)) takes the form:

$$\dot{\psi}_j=\frac{\partial \mathscr{H}_R}{\partial J_j}, \qquad \dot{J}_j=-\frac{\partial \mathscr{H}_R}{\partial \psi_j} \tag{A 2}$$

with the Hamiltonian (10) where $f_{0,1,2,3}(J_1, J_2)$ are determined by the expressions

$$\left.\begin{aligned}
f_0(J_1,J_2)&=(J_1+J_2)^2+(J_1+J_2+J_4)^2+(J_3-J_1)^2+J_2^2\\
&\quad+(J_6-J_2)^2+(J_5-J_1)^2;\\
f_1(J_1,J_2)&=8[(J_1+J_2)(J_1+J_2+J_4)(J_3-J_1)(J_5-J_1)]^{1/2};\\
f_2(J_1,J_2)&=8[(J_1+J_2)(J_1+J_2+J_4)(J_2-J_6)J_2]^{1/2};\\
f_3(J_1,J_2)&=8[(J_3-J_1)(J_5-J_1)(J_2-J_6)J_2]^{1/2}.
\end{aligned}\right\} \tag{A 3}$$

Hamiltonian $\mathscr{H}_R$ is independent of variables ψ_δ ($\delta=3,4,5,6$). This indicates that the values J_δ canonically conjugate to ψ_δ are motion integrals of system (A 2), (10). Integrals (5), (8) of system (3) are linear combinations of J_δ:

$$L=2(J_3+J_4+J_5+J_6), \qquad R_1=2J_4, \qquad R_2=2(J_5-J_3), \qquad R_3=2J_6. \tag{A 4}$$

Thus the initial system (3) has been reduced to the Hamiltonian system (9) (compare with (A 2)) and (10) with two degrees of freedom ψ_n, J_n ($n=1,2$), with the dynamics independent of the motions associated with the degrees of freedom ψ_i, J_i ($i=3,4,5,6$).

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