Optical Cavities

Cavity provides the feedback system needed for laser operation!

A cavity mode is a field distribution that reproduces itself in relative shape and in relative phase after a round trip.

To follow a Gaussian beam through a round trip in a cavity we employ the ABCD law.

1. We assume that the Hermite-Gaussian beams are the characteristic modes of the optical cavity (Mirrors exactly match the surfaces of constant phase of the beams.)

2. The complex beam parameter repeats itself after a round trip!

\[ q(z_1 + \text{roundtrip}) = q(z_1) \]
\( q(z_1 + \text{roundtrip}) = q(z_1) \)

\[
q(z_1) = \frac{A q(z_1) + B}{C q(z_1) + D}
\]

\[
A \begin{pmatrix} qz \\ qz \end{pmatrix} + B = C \begin{pmatrix} qz \\ qz \end{pmatrix} + D
\]

\( Cq^2 + Dq = Aq + B \)

\[
B \left( \frac{1}{q} \right)^2 + 2 \left( \frac{A - D}{2} \right) \left( \frac{1}{q} \right) - C = 0
\]

\[
\frac{1}{q} = -\frac{A - D}{2B} \pm \frac{1}{B} \left[ \left( \frac{A - D}{2} \right)^2 + BC \right]^{\frac{1}{2}}
\]

\[
AD - BC = 1 \quad \frac{A^2}{4} - \frac{AD}{2} + \frac{D^2}{4} + BC \left( 1 - \frac{AD}{2} + \frac{AD}{2} \right) = \left( \frac{A + D}{2} \right)^2 - 1
\]

\[
\frac{1}{q} = -\frac{A - D}{2B} - j \left[ \frac{1 - \left( \frac{A + D}{2} \right)^2}{B} \right]^{\frac{1}{2}}
\]
\[
\frac{1}{q} = \frac{1}{R} - j \frac{\lambda}{\pi w^2}
\]

\[
R(z_1) = -\frac{2B}{A-D}
\]

\[
\frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}}
\]

\[
T = \begin{pmatrix}
1 & d + z_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{f} & \frac{d-z_1}{f} \\
1 & \frac{d-z_1}{f}
\end{pmatrix}
\]

If we start the unit cell at the flat mirror \((z = 0)\)

\[
T = \begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{f} & \frac{d}{f} \\
\frac{1}{f} & \frac{d}{f}
\end{pmatrix}
\begin{pmatrix}
1 - \frac{d}{f} & d + d\left(1 - \frac{d}{f}\right) \\
0 & 1 - \frac{d}{f}
\end{pmatrix}
\]

\[
\pi w_0^2 = \frac{d + d\left(1 - \frac{d}{f}\right)}{\left[1 - \left(1 - \frac{d}{f}\right)^2\right]^{1/2}} = \frac{2d\left(1 - \frac{d}{R}\right)}{\left[4\left( \frac{d}{R}\right)\left(1 - \frac{d}{R}\right)\right]^{1/2}} \quad \text{for} \quad f = \frac{R}{2}
\]

\[
\pi w_0^2 = \left(dR\right)^{1/2}\left(1 - \frac{d}{R}\right)^{1/2}
\]
\[ w^2(z) = w_0^2 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right] \]
\[ z_0^2 = \left( \frac{\pi w_0^2}{\lambda} \right)^2 \]
\[ \frac{\pi w^2(d)}{\lambda} = (dR)^{\frac{1}{2}} \left( 1 - \frac{d}{R} \right)^{\frac{1}{2}} \frac{1}{dR} \left[ 1 + \frac{d^2}{(1 - \frac{d}{R})^2} \right] = \left( \frac{dR}{R} \right)^{\frac{1}{2}} \]

On a spherical mirror

\[ \frac{\lambda \sqrt{d R_2}}{\pi} \]

(Not physical)

Gaussion beam analysis does not apply

Flat mirror

Spot size on the spherical mirror

\[ \frac{\pi w_0^2}{\lambda} = (dR)^{\frac{1}{2}} \left( 1 - \frac{d}{R} \right)^{\frac{1}{2}} \]

\[ w_0 \text{ becomes imaginary} \]
The requirement that mirrors must match the surface of the constant phase (so that a cavity mode is excited) allows one to find where the mirrors with $R_1$ & $R_2$ should be placed, in general case.

1) $z_1 + z_2 = d$  
2) $R(z_2) = R_2 = z_2 \left[1 + \left(\frac{z_0}{z_2}\right)^2\right]$  
3) $R(z_1) = -R_1 = -z_1 \left[1 + \left(\frac{z_0}{z_1}\right)^2\right]$

The wave front on the left at $z=0$ has a (mathematically) negative radius of curvature, but we know that the mirror $R_1$ has positive (focusing) properties. We treat all distances $z_1, z_2$ as positive numbers and let the radii of curvature of the mirrors carry their own sign.

Solving (involving algebra)....

$$z_0^2 = \left(\frac{\pi w_0^2}{\lambda}\right)^2 = \frac{d(R_1-d)(R_2-d)(R_1+R_2-d)}{(R_1+R_2-2d)^2}$$

$$z_1 = \frac{d(R_2-d)}{R_1+R_2-2d} \quad z_2 = \frac{d(R_1-d)}{R_1+R_2-2d}$$

$$R_1 = \infty \quad z_0^2 = d(R_2-d) \Rightarrow dR_2 \left(1 - \frac{d}{R_2}\right) \quad z_1 = 0; z_2 = d \quad \frac{\pi w_0^2}{\lambda} = (dR)^{1/2} \left(1 - \frac{d}{R}\right)^{1/2}$$

Same result
Summary

1) Postulate that Hermite-Gaussian Beams are the normal modes for the cavity.

2) Formulate an equivalent transmission system for this cavity showing one round trip. Identify a unit cell.

3) Force the complex beam parameter to transform into itself after a round trip by use of the ABCD law.

4) Evaluate \( R \) and \( w \) using:

\[
R(z) = -\frac{2B}{A-D}
\]

\[
\frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}}
\]

The theory applies for Stable cavity only!
\[
E_0^2 V = \int_0^d dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy E(x, y, z) E^*(x, y, z)
\]

\[
E_0^2 V_{m,n} = E_0^2 \int_0^{\infty} \frac{w_0^2}{w^2(z)} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy H_m^2 \left( \frac{\sqrt{2x}}{w} \right) e^{-\frac{2x^2}{w^2}} \times H_n^2 \left( \frac{\sqrt{2y}}{w} \right) e^{-\frac{2y^2}{w^2}}
\]

use \[ u = \frac{\sqrt{2x}}{w} \text{ or } \frac{\sqrt{2y}}{w} \]

\[
V_{m,n} = \int_0^d \frac{w_0^2}{2} dz \left[ \int_{-\infty}^{\infty} H_m^2 (u) e^{-u^2} du \right] \left[ \int_{-\infty}^{\infty} H_n^2 (u) e^{-u^2} du \right] 2^n n! \sqrt{\pi}
\]

\[
V_{m,n} = \frac{\pi w_0^2}{2} d \left( m! n! 2^{m+n} \right)
\]

Area \times length \quad \text{Modification for high-order modes}
Example (textbook)

\[ V_{0,0} = 1.38 \, \text{cm}^3 \]

\[ = \frac{\pi w_0^2}{2} \times d \]

He-Ne laser

\[ P = 0.1 \, \text{torr} \, \text{(of neon)} \]

Each atom is excited (by the gas discharge) and thus producing a photon at 632.8 \, \mu m, say, 10 times per second.

Energy per photon

\[ h\nu = \frac{hc}{\lambda} = 3.14 \times 10^{-19} \, J = 1.96eV \]

\[ \times \, \# \, \text{of Ne atoms} \, = \, 0.1(3.54 \times 10^{16})V_{0,0} = 4.88 \times 10^{15} \]

\[ \times \, \text{(average excitation per atom = average emission per atom)} \, = \, 10 \, \text{sec}^{-1} \]

\[ = \, \text{Power} \, = \, 15.3mW \]

typical!