ME614 Spring 2017 - Homework 1

Spatial Discretization

(Due February 6, 2017)

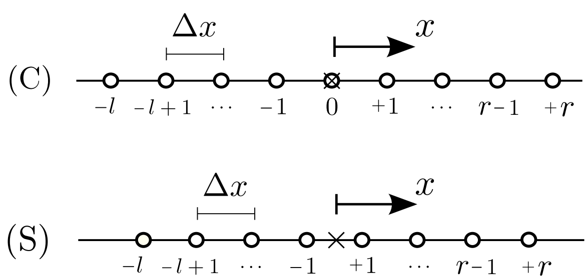
Please prepare this homework in the folder `/homework1/` of your git repository and submit it following the instructions in the last section of Homework 0. The main script should be named `homework.py` and placed in the `/homework1/code/` folder. The submitted code needs to run and create all of the required plots in the folder `/homework1/report/figures/` at once. The use of LaTeX for your report is strongly recommended (but not required) and you can start with a template from http://www.latextemplates.com/. Discussions and sharing of ideas are encouraged but individually prepared submissions (codes, figures, written reports, etc.) are required. A plagiarism detection algorithm will be run against all codes submitted. Do NOT include in your git repository files that are not required for homework submission (e.g. the syllabus, zip files with python libraries, other PDFs, sample python sessions etc). Follow exactly the suggested folder structure of your homework git repository as all of the grading tools on the instructor's end are scripted. Points will be deducted from late submissions at a rate of 20% of the overall homework value per day late.

Problem 1

With the use of polynomial fitting, consider the numerical approximation of the first derivative of the function

$$f(x) = \tanh(x)\sin(5x + 1.5) \tag{1}$$

at $x = 0$ (indicated with an '$\times$') with the uniformly spaced computational stencils shown below.



	centered	biased
(C)	$l = r = 1$	$l = 0,\ r = 1$
(C)	$l = r = 2$	$l = 0,\ r = 2$
(C)	$l = r = 3$	$l = 0,\ r = 3$
(S)	$l = r = 1$	$l = 1,\ r = 1$
(S)	$l = r = 2$	$l = 1,\ r = 2$
(S)	$l = r = 3$	$l = 1,\ r = 3$

The polynomial interpolant should exploit the full number of points (indicated with circles), that is $N = l + r + 1$ for the (C) arrangement, and $N = l + r$ for the (S) arrangement, for every given value of l and r. Please perform the following tasks:

(a) Plot in log-log scale the absolute value of the truncation error, ϵ, versus the inverse of the grid spacing, Δx^{-1}, as the grid is refined ($\Delta x \to 0$), starting with a unitary spacing ($\Delta x = 1.0$). Compare results among all combinations of collocated (C) and staggered (S) arrangements, and centered and biased reconstructions shown in the table above[1]. Plot with a thin dashed line the reference truncation error $\epsilon_{\text{ref}} \propto \Delta x^n$ for various values of n (order of accuracy). Make sure to label the line on the plot with the corresponding value of n.

[30%]

(b) Does the order of accuracy always correspond to the order of the polynomial interpolant? For a given polynomial order, p, what is the minimum and maximum order of accuracy you can achieve when evaluating numerically the first derivative? Briefly discuss by relying on your numerical results from the previous task.

[5%]

[1] The biased case for $l = r = 1$ is not technically biased but it is convenient to define it for scripting purposes.

Problem 2

Perform the following numerical tasks adopting a uniform grid with N points:

(a) Generate discrete numerical operators in the form of sparse matrices corresponding to the first and third derivatives, with third-order and fifth-order *polynomial* reconstructions (for a total of 4 different operators). With the `spy` plot function of Python or Matlab plot the resulting matrix pattern for $N = 10$.

[20%]

(b) Solve the following one-dimensional boundary value problem (BVP) in the domain $[x_0, x_1]$:

$$\frac{d^3}{dx^3}u(x) = f^{(3)}(x) \tag{2}$$

with,

$$\begin{aligned} \widehat{u}_0 &= f(x)|_{x=x_0} \\ \left.\widehat{\frac{du}{dx}}\right|_0 &= \left.f^{(1)}(x)\right|_{x=x_0} \\ \left.\widehat{\frac{du}{dx}}\right|_{N-1} &= \left.f^{(1)}(x)\right|_{x=x_1} \end{aligned} \tag{3}$$

where $f^{(3)}(x)$ is the (non-trivial) third derivative of an arbitrary function, $f(x)$, of your choice, and 0 and $N-1$ are the first and last points of the computational domain, corresponding to x_0, x_1, which are also arbitrary. Plot the root-mean-square (RMS) of the error[2], $\epsilon = |\hat{u} - f|$, against the inverse of the grid spacing, Δx^{-1} for the two polynomial orders mentioned in part (a).

[25%]

Problem 3

Solve Problem 2(b) using an n-th order Padé scheme, where n is an appropriate order for the BVP above (e.g. $n = 3$ might not be enough to solve the requested problem).

(a) Show analytical derivations for the specific Padé scheme adopted. Note that the adopted scheme needs to be able to discretize a third derivative in space.

[10%]

(b) Once again, plot the root-mean-square of the the error, $\epsilon = |\hat{u} - f|$, where u now corresponds to the solutions obtained using the adopted Padé scheme, against the inverse of the grid spacing and show that you have achieved the suggested n-th order of accuracy.

[10%]

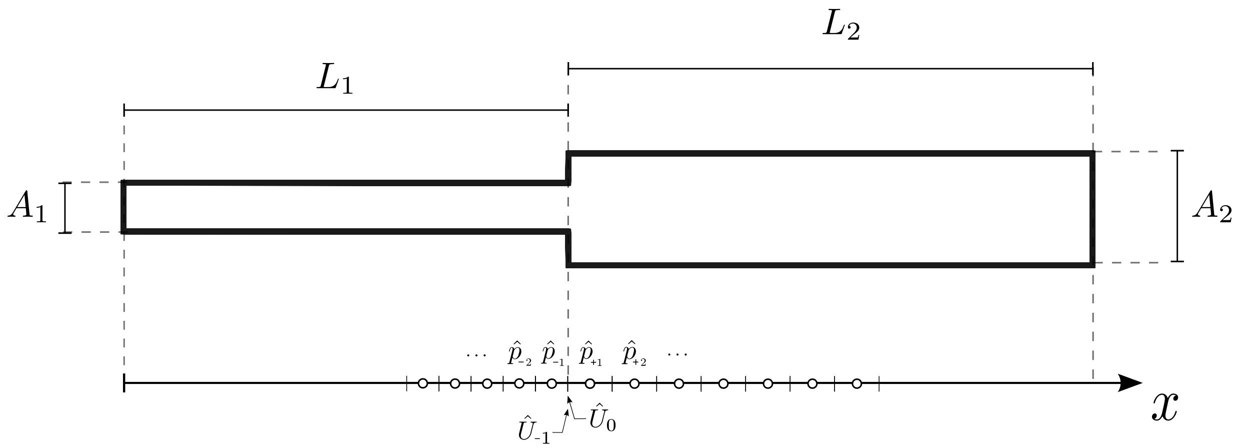
For more details on compact finite difference schemes, refer to (Lele 1992).

[2]The root-mean-square is the square root of the integral average (or arithmetic average in this case) of the squares of the local value of the error. Therefore, you need to make sure you average the square of the local truncation error *first* and then take the square root.

Bonus Questions

You can only pick one of the bonus problems below. Note that you don't need to solve a bonus problem to get 100% on this homework. Each bonus question is worth extra +5% of the total homework value and may be used to replace any items above that have not been (intentionally or non intentionally) addressed. Bonus questions may also be used simply as extra credit, which will count towards the final grade.

Problem A



Isentropic linear acoustic wave propagation in a variable area duct $A(x)$ is governed by the following equations in the time-domain:

$$\frac{d}{dt}p' = -\frac{\rho_0 a_0^2}{A}\frac{dU'}{dx} \tag{4a}$$

$$\frac{d}{dt}U' = -\frac{A}{\rho_0}\frac{dp'}{dx} \tag{4b}$$

where p' and U' are the instantaneous pressure and volumetric flow rate fluctuations, A is the cross-sectional area available to the gas flow, ρ_0 is the base density, and $a_0 = \sqrt{\gamma R T_0}$ is the speed of sound calculated from the based temperature distribution, T_0. The base pressure, P_0 is assumed to be constant, and base properties satisfy the relation $P_0 = \rho_0(x)\, R\, T_0(x)$.

Solve numerically the eigenvalue problem resulting from assuming normal modes for all fluctuating quantities with the convention $e^{+i\omega\, t}$ where ω is the angular frequency. Consider the geometry shown in figure composed by two constant-area segments with area ratio $A_1 = A_2/2$ and lengths $L_1 = L_2 = 1$ m, at atmospheric conditions. Discretize the governing equation with a centered second-order a staggered variables approach where (4a) is written for the pressure locations (circles in figure) and (4b) is written for the flow rate locations (vertical bars in figure).

This pressure-volume-flow-rate formulation allows for the straightforward imposition of boundary conditions. The continuity of pressure (but not of its derivative) at the interface of sections with different cross-sectional areas is enforced by linear extrapolation (Lin, et al. 2016):

$$-\hat{p}_{-2} + 3\hat{p}_{-1} = 3\hat{p}_{+1} - \hat{p}_{+2} \tag{5}$$

$$\hat{U}_{-1} = \hat{U}_1 \tag{6}$$

where -2 and -1 indicate the second-last and last point of a segment and +1 and +2 indicate the first and second point of the following segment.
Plot the magnitude of the pressure and velocity eigenfunction for the first and second acoustic mode.

Problem B

Consider the following filtering operation

$$\overline{f}(x) = \int_{-\Delta}^{\Delta} f(x-\xi)G(\xi;x)d\xi \tag{7}$$

where $f(\cdot)$ is a "noisy" function defined on an interval of your choice. Generate $f(\cdot)$ by superimposing random Gaussian noise to a coherent signal (e.g. a sine wave). Design the filter kernel $G(\xi;x)$ to be variable in space x, filtering more aggressively for higher values of x. Construct the operator $\mathbf{F}$, representing the discrete equivalent of (7), and the operator $\mathbf{D}$, representing the first derivative on the chosen mesh and show that $\mathbf{F}\cdot\mathbf{D} \neq \mathbf{D}\cdot\mathbf{F}$. For more details on the relevance of this exercise for turbulent simulations see Vasilyev, et al. (1998).

References

S. K. Lele (1992). 'Compact finite difference schemes with spectral-like resolution'. *Journal of Computational Physics* **103**(1):16–42.

J. Lin, et al. (2016). 'High-fidelity simulation of a standing-wave thermoacoustic–piezoelectric engine'. *Journal of Fluid Mechanics* **808**:19–60.

O. V. Vasilyev, et al. (1998). 'A general class of commutative filters for LES in complex geometries'. *Journal of Computational Physics* **146**(1):82–104.