Universal Scaling of Acoustic and Thermoacoustic Waves in Compressible Fluids

Mario Tindaro Migliorino and Carlo Scalo
School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907, USA
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We have derived the set of reference scaling parameters yielding collapse of isentropic acoustic and thermoacoustic (or heat-release-induced) waves across different pure compressible fluids with an assigned equation of state. The resulting reference pressure and velocity are consistent with classic acoustic scaling. The reference temperature and heat release rate need to be expressed in terms of the isobaric thermal expansion coefficient $\alpha_{p0}$, in order to ensure collapse of all thermo-fluid-dynamic fluctuations. The proposed scaling is extended to non-isentropic waves and verified against data from highly-resolved one-dimensional Navier-Stokes simulations. Conditions tested include freely propagating isentropic acoustic waves and thermoacoustic compression waves up to shock strength of 4.27, for six different supercritical fluids each in states ranging from compressible liquid to near-ideal gas, and spanning seven orders of magnitude of variation in the imposed heat release rate.

Waves in compressible fluids are propagating disturbances affecting all thermo-fluid-dynamic variables. However, common practice in acoustics is to use, when possible, pressure and velocity fluctuations (or pressure only) as the sole working variables [1]. This choice is consistent with the fundamental nature of sound waves, i.e. self-propagating patterns of compressions and dilatations, inducing and induced by spatial gradients in particle displacements or velocities. No other variables but pressure and velocity are thus needed to intuitively understand sound propagation and the mechanical or acoustic power associated with it.

As a result, a commonly adopted approach towards the derivation of a dimensionless set of governing equations has traditionally been focused on collapsing pressure and velocity fluctuations only, with little consideration given to other fluctuations, such as temperature, enthalpy or internal energy, which are still present and equally relevant [2]. Temperature fluctuations for example are of primary importance in studies of, to cite a few, thermophones [3], sound interaction with thermal boundary layers [4], thermoacoustic energy conversion [5][7], and thermoacoustic (or heat-release-induced) waves [5][9][10]. Moreover, thermoacoustic-wave-induced temperature fluctuations are the governing mechanism, referred to as the Piston Effect, for thermal relaxation in near-critical fluids in enclosed cavities [14][20].

The identification of the correct set of scaling parameters is hence required to ensure universal applicability of a dimensionless set of governing equations. To the authors’ knowledge, there have been no prior attempts towards developing an appropriate dimensionless scaling strategy able to provide a unified description of linear waves across different fluids and in different states, ranging from compressible liquids to ideal gases. For example, as shown below, using the fluid’s base density and speed of sound to scale pressure and velocity fluctuations, and the base temperature to scale temperature fluctuations [21], does not yield dimensionless collapse of temperature fluctuations, even in the simple case of ideal gases.

In the first part of this Letter we address this knowledge gap by deriving a set of reference scaling parameters for all fluctuations in the thermo-fluid-dynamic variables, demonstrating full collapse of isentropic acoustic and thermoacoustic planar waves across different fluids, each taken at supercritical conditions ranging from pseudo-liquid to pseudo-gaseous. In the second part of the Letter we extend the derived scaling to the jumps in thermo-fluid-dynamic variables across thermoacoustic shock waves up to shock strength of 4.27, accounting for the effects of non-isentropic wave propagation by introducing a new effective ratio of specific heats.

To verify the proposed scaling, highly-resolved single-phase high-order fully compressible Navier-Stokes simulations are performed with the solver Hybrid [22] for six different fluids at supercritical pressure conditions $p^*_0 = 1.1 p^*_c$ (table I). The base temperature $T^*_0$ is varied to achieve a range of conditions including: pseudo-liquid (PL), pseudo-boiling (PB), pseudo-gaseous (PG), modeled with the Peng-Robinson [23] equation of state (EoS) and Chang’s method [24][25] and near-ideal gas (IG), modeled as a perfect gas and with Sutherland’s law. This Letter contains data from a total of 187 simulations.

<table>
<thead>
<tr>
<th>state</th>
<th>$T^*_0$</th>
<th>CO$\textsubscript{2}$</th>
<th>O$\textsubscript{2}$</th>
<th>N$\textsubscript{2}$</th>
<th>CH$\textsubscript{3}$OH</th>
<th>R-134a</th>
<th>R-218</th>
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<tbody>
<tr>
<td>PL</td>
<td>0.89</td>
<td>★</td>
<td>▲</td>
<td>▽</td>
<td>■</td>
<td>★</td>
<td></td>
</tr>
<tr>
<td>PB</td>
<td>1.02</td>
<td>★</td>
<td>▲</td>
<td>▽</td>
<td>■</td>
<td>★</td>
<td></td>
</tr>
<tr>
<td>PG</td>
<td>1.11</td>
<td>★</td>
<td>▲</td>
<td>▽</td>
<td>■</td>
<td>★</td>
<td></td>
</tr>
<tr>
<td>IG</td>
<td>2.20</td>
<td>★</td>
<td>▲</td>
<td>▽</td>
<td>■</td>
<td>★</td>
<td></td>
</tr>
<tr>
<td>$p^*_c$ (MPa)</td>
<td>7.3773</td>
<td>5.043</td>
<td>3.398</td>
<td>8.097</td>
<td>4.059</td>
<td>2.88</td>
<td></td>
</tr>
<tr>
<td>$T^*_c$ (K)</td>
<td>304.13</td>
<td>154.58</td>
<td>126.2</td>
<td>512.64</td>
<td>374.26</td>
<td>345.1</td>
<td></td>
</tr>
</tbody>
</table>

TABLE I. Marker legend for six different fluids at base state conditions ranging from pseudo-liquid to ideal gas. All cases are considered at $p^*_0 = 1.1 p^*_c$. Values of fluid-specific critical pressures $p^*_c$ and temperatures $T^*_c$ are also reported.

We start from the simple case of planar isentropic acoustic waves evolving in a uniform quiescent base state, hereafter indicated with the subscript ‘0’. In this case pressure $\delta p^*$ and velocity $\delta u^*$ fluctuations are the only...
flow parameters needed to completely characterize the state of the wave. The proper and commonly used normalization choice in this case is

$$\delta p = \frac{\delta p^*}{\rho_0^* a_0^*}, \quad \delta u = \frac{\delta u^*}{a_0^*}, \quad x = \frac{x^*}{t^*}, \quad t = \frac{t^*}{t^*/a_0^*}, \quad (1)$$

where $\rho_0^*$ and $a_0^*$ are the base density and isentropic speed of sound, and $x^*$ and $t^*$ the independent spatial and temporal coordinates, and $t^*$ a reference length scale; all dimensional quantities are denoted by the superscript $(\ast)$, which is omitted in their dimensionless counterpart. Applying the normalization in Eq. (1) to the linearized continuity and momentum equations, assuming isentropic flow, yields

$$\frac{\partial}{\partial t} \delta u = -\frac{\partial}{\partial x} \delta p, \quad \frac{\partial}{\partial t} \delta p = -\frac{\partial}{\partial x} \delta u, \quad (2)$$

whose solution in an unbounded domain can be expressed without loss of generality in the self-similar form

$$\delta u_{\pm} = \pm \delta p_{\pm} = f_{\pm}(\xi_{\pm}). \quad (3)$$

The two functions $f_{\pm}()$ of the traveling-wave coordinate $\xi_{\pm} = x \mp M_s t$ (with phase speed $M_s = a_0 = 1$ in this case) can be independently and arbitrarily assigned provided $\max_{\xi} |f(\xi)| \ll 1$ to respect isentropicity. The dependency on the base state, and hence also on the specific fluid properties, of the governing equations (2) has been completely absorbed by the normalization (1). The steps leading to Eq. (2) and Eq. (3) do not require the specification of an equation of state (EoS) nor of an explicit normalization for temperature fluctuations $\delta T^*$, however, they do entail the normalization $\delta \rho = \delta \rho^*/\rho_0^*$. The scaling of freely propagating isentropic planar waves will hereafter be focused on, without loss of generality, only on right-propagating waves ($\delta u = \delta p = \delta u_+ = \delta p_+, \xi = \xi_+$).

We start by noticing how the commonly adopted normalization of temperature fluctuations $\delta T^*$, using the base state temperature $T_0^*$ as a reference, does not collapse isentropic temperature perturbations, associated to the same acoustically scaled waveform $f(\xi)$, across different fluids, even in ideal gas conditions (figure 1(a)). This issue is resolved by noticing that an isentropic fluctuation of a generic quantity $\delta \varphi^*$ can be expressed as a sole function of pressure fluctuations via evaluation of the thermodynamic derivative $\delta \varphi^*/\delta p^* = \partial \varphi^*/\partial p^*|_{s^*,\ast,0}$, yielding

$$\delta T^* = \frac{\alpha_0^* T_0^*}{\rho_0^* c_0^*} \delta p^*, \quad \delta h^* = \frac{1}{\rho_0^*} \delta p^*, \quad \delta e^* = \frac{\rho_0^*}{\rho_0^* a_0^*} \delta p^*, \quad (4)$$

where $\delta h^*$ and $\delta e^*$ are the specific (per unit mass) enthalpy and internal energy fluctuations, $c_0^*$ is the isobaric specific thermal capacity and

$$\alpha_0^* = -\rho_0^{* -1} \partial \varphi^*/\partial T^*|_{p^*,\ast,0} \quad (5)$$

is the isobaric thermal expansion coefficient, both calculated at base state conditions. Applying the relation $a_0^* T_0^* \alpha_0^* \rho_0^* / c_0^* = \gamma_0 - 1$, where $\gamma_0 = c_0^*/\rho_0^*$ is the ratio of specific isobaric and isochoric thermal capacities, and the normalization in Eq. (1), to Eq. (4), the correct normalization achieving the desired collapse (figure 1(b)) reads

$$\delta T = \frac{\alpha^*_{p_0} T^*}{\gamma_0 - 1} \delta T^*, \quad \delta h = \frac{1}{\alpha_0^*} \delta h^*, \quad \delta e = \frac{\gamma_0}{\alpha_0^*} \delta e^*, \quad (6)$$

where $\gamma_0 = \rho_0^* a_0^* / \alpha_0^*$ is the isentropic exponent (6).

Equations (1) and (6) define the complete set of reference scaling parameters $\varphi^*_{ref}$, summarized in table I, for isentropic wave propagation in a generic compressible fluid, collapsing all thermo-fluid-dynamic fluctuations across different fluids, and, incidentally, also among themselves,

$$\delta u = \delta p = \delta \rho = \delta T = \delta h = \delta e = f(\xi), \quad (7)$$

which is a direct result of the single degree of thermodynamic freedom.

Normalizing temperature, internal energy, and enthalpy fluctuations using the base temperature $T_0^*$ only

FIG. 1. Scaled dimensionless temperature fluctuations from inviscid computations of a right-traveling acoustic wave for $f(\xi) = 10^{-6} \sin(\xi)$ (see Eq. (3)) for all fluids and conditions in table I. (a) Non-universal scaling based on base temperature; (b) scaling only valid for IG; (c) proper universal scaling (table II). Same results are obtained for other variables such as $\delta e$, $\delta h$, and $\delta \rho$ (not shown).
yields collapse across different ideal gases with the same value of $\gamma_{0}$, since in that case the proposed scaling parameters revert to $\delta T^*|_{IG} = (\gamma_{0} - 1)T^*$, $\delta e^*|_{IG} = R^*T^*$, $\delta h^*|_{IG} = \gamma_{0}R^*T^*$, where $R^*$ is the gas constant. Ideal gas temperature perturbations made dimensionless only via $T_{0}^*$ (figure 1a), in fact, do not collapse unless $\gamma_{0} - 1$ is also taken into consideration (figure 1b). Finally, full collapse across all fluids and in all conditions (figure 1c) can only be achieved employing the correct reference temperature $\delta T^* = (\gamma_{0} - 1)/\alpha_{p0}^*$ (figure 2). The same result is obtained for enthalpy and internal energy fluctuations by using their respective scaling parameters (not shown).

$k^* = k^*(\rho^*, T^*)$ is the thermal conductivity both modeled with Chung’s method \cite{21, 24} for PL, PB, PG conditions. Finally, $\dot{Q}^* [W/m^3]$ is the volumetric heat release rate, expressed as

$$\dot{Q}^*(x^*, t^*) = \Omega^*g^*(x^*), \quad g^*(x^*) = \frac{1}{\ell^*\sqrt{2\pi}}e^{-\frac{(x^*/\ell^*)^2}{2}},$$

where $\Omega^* [W/m^2]$ is the planar heat release rate and $g^*(x^*) [m^{-1}]$ is a Gaussian function with unitary (non-dimensional) integral on the real axis, with characteristic width $\ell^* = 0.75\mu m$, inspired by the experiments of Miura et al. \cite{27}. Simulations are carried out only for $x^* \geq 0$, with adiabatic wall conditions imposed at $x^* = 0$, and halted before perturbations reach the right boundary. The computational domain is sufficiently long to allow shocks to reach conditions of inviscid equilibrium propagation. The shocks are resolved with a minimum number of 14 grid points, corresponding to a grid spacing of $\Delta x^* = 0.04 \mu m$, and with a maximum time step of $\Delta t^* = 0.01 ns$. Simulations are performed for values of $\Omega^*$ reported in table III.

$$\Omega^* | \begin{array}{cccc}
10^5 & 10^7 & 10^9 & (1,3,6) \times 10^{10} \\
(1,3,6) \times 10^{11} & (1,3,6) \times 10^{12} & 10^{12}
\end{array}$$

TABLE III. Planar heat release rates [W/m²] used in numerical simulations of thermoacoustic waves. Values of the order of $10^{11}$ (′) are used only for PL conditions, and of $10^{12}$ (″) only for CH₃OH in PL conditions.

For sufficiently low heat release rates (quasi-isentropic regime), the amplitude of the generated compression waves can be predicted via

$$\Pi^* = p^*_1 - p^*_0 = \frac{\partial p^*}{\partial s^*} |_{p^*, \sigma^*} \Sigma^* = \frac{a_0^*\alpha_{p0}^*}{c_{p0}^*} \Omega^*/2,$$

where $\partial p^*/\partial s^* |_{p^*, \sigma^*} = \rho_0^*T_0^*\alpha_{p0}^*a_0^*\gamma_0^*/c_{p0}^*$, and $\Sigma^* = \Omega^*/(2\rho_0^*\alpha_{p0}^*T_0^*)$, is the reversible dimensional entropy jump, and the subscript ‘1’ indicates the post-compression state. Eq. (11) is consistent with the formula given by Miura et al. \cite{27} and should be made dimensionless as follows:

$$\Pi = \frac{\Pi^*}{\delta p^*_ref} = \frac{\Omega^*/2}{\rho_0^*\alpha_{p0}^*a_0^*} = \Omega,$$

where $\Omega^*$ is normalized with $\Omega^*_{ref}$ (figure 3, table II), and $\Pi$ is the shock strength \cite{28}, much smaller than one for quasi-isentropic compressions. Under such conditions, all the thermo-fluid-dynamic jumps are only a function of $\Omega$, consistently with Eq. (7).

Analogously to figure 1, figure 4 shows how previously adopted normalizations of heat release rate for thermoacoustic waves \cite{29} lead to no (figure 4a) or partial (figure 4b) collapse, ultimately only achieved by the proposed universal scaling (figure 4c).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\Omega^*$ & $10^5$ & $10^7$ & $10^9$ \\
\hline
(1,3,6) $\times 10^{10}$ & (1,3,6) $\times 10^{11}$ & $10^{12}$ \\
\hline
\end{tabular}
\caption{Planar heat release rates [W/m²] used in numerical simulations of thermoacoustic waves. Values of the order of $10^{11}$ (′) are used only for PL conditions, and of $10^{12}$ (″) only for CH₃OH in PL conditions.}
\end{table}
FIG. 3. Heat release rate reference scaling parameter versus reduced temperature for all fluids and conditions in table I.

FIG. 4. Scaled pressure fluctuations of quasi-isentropic thermoacoustic waves for dimensional heat release rate $\Omega^* = 10^5 \text{W/m}^2$. (a) Non-universal scaling (Chu [29]); (b) scaling only valid for IG; (c) correct universal scaling (table II).

In the regime governed by reversible heat addition, Eq. (12) provides a near-isentropic, low-wave amplitude prediction and full collapse of data from the numerical simulations across all fluids and conditions in tables II and III, confirming the universality of the proposed scaling for $\Omega < 10^{-1}$ (figure 3). While the extraction of pressure jumps $\Pi^*$ from the numerical simulation data is trivial in such regime due to the flatness of the post-compression state (figure 4), for larger wave amplitudes the latter is identified as the location of completed coalescence of the compression characteristics, pinpointing the start of the inviscid equilibrium propagation phase, where viscous and conductive effects are not affecting the compression amplitude.

Thermoacoustic wave amplitudes depart from the prediction in Eq. (12) for $\Omega > 10^{-1}$ (see insets of figure 5) following a necessarily fluid-specific and condition-specific law. Shock-amplitude prediction is still possible in this regime by introducing an effective ratio of specific heats $\gamma_\pi$ defined based on the relation

$$\Omega = \sqrt{2\Pi(\gamma_\pi\Pi + 1)}(2 + (\gamma_\pi + 1)\Pi)^{-0.5},$$  \hspace{1cm} (13)

obtained by extending the parametrization originally derived by Chu [29] for an inviscid ideal gas to a generic fluid adopting the proposed scaling (table II). For IG conditions, $\gamma_\pi$ reverts to $\gamma_0$. Moreover, Eq. (13) reverts to Eq. (12) for $\Pi \to 0$, removing the dependency from $\gamma_\pi$, consistently with the single degree of thermodynamic freedom intrinsic to isentropic or near-isentropic waves. When non-reversible entropy generation occurs across the thermoacoustic compression, the specification of the fluid-specific and state-specific $\gamma_\pi$ (second degree of thermodynamic freedom) is necessary to predict the pressure jump. Values of $\gamma_\pi$ have been fitted against the data from the numerical simulations via Eq. (13) (table IV) demonstrating formidable consistency (figure 6) up to shock strengths of $\Pi = 4.27$.

In summary, we have derived the correct set of reference scaling parameters able to achieve collapse of
isentropic acoustic and thermoacoustic waves across different pure compressible fluids in conditions ranging from liquid-like to gas-like. Data from highly resolved one-dimensional Navier-Stokes numerical simulations has been adopted to verify the effectiveness of proposed scaling strategy and to aid its extension to the non-isentropic regime, where thermoacoustic shock waves have been investigated. We have demonstrated that in this regime a new effective ratio of specific heats, $\gamma_\pi$, needs to be specified to enable prediction of jumps of the thermo-fluid-dynamic variables.

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* migliom@purdue.edu