Real fluid effects on thermoacoustic standing-wave resonance in supercritical CO$_2$

Mario Tindaro Migliorino,* Prateek Gupta† and Carlo Scalo‡

Purdue University, West Lafayette, IN 47907, USA

We have investigated real fluid effects in a canonical thermoacoustically unstable duct filled with supercritical CO$_2$ at a base pressure of $p_0 = 1.1 p_{cr}$, where $p_{cr} = 7.3773$ MPa is the critical pressure of CO$_2$. Thermodynamic and transport properties of CO$_2$ are obtained from the Peng-Robinson equation of state and Chung’s model. A toy thermoacoustically unstable resonator is investigated, composed of an inviscid hot cavity filled with pseudo-gas ($T_{hot} > T_{cr}$), a thermoacoustic stack with an imposed axial wall-temperature gradient via isothermal boundary conditions, and an inviscid resonator ($T_{cold} < T_{cr}$) filled with pseudo-liquid. The parameter $T_{cold}/T_{cr}$ is varied from 0.65 to 0.9 such that total 6 computational cases considered retain the pseudo-boiling conditions inside the stack. The acoustic energy budgets are derived theoretically, showing that the thermoacoustic production is proportional to $\Theta = d \ln \rho_0^{-1}/dx$ in the standing wave configuration, where $\rho_0$ is the base state density. Due to real fluid effects, $\Theta$ spikes at spatial location of PB thus rendering thermoacoustic instability sensitive to PB. Maximum growth rates are achieved corresponding to a location of the pseudo-interface at approximately the center of the stack. Moreover, at the pseudo-interface, the spatial derivative of the base acoustic impedance spikes causing discontinuities in the spatial derivatives of the eigenmodes. Moreover, changes in the Prandtl number ($\text{Pr}$) result in increased thermoacoustic production of energy density due to traveling wave component for small $h/2h_0$. Ongoing work includes high-order fully compressible numerical simulations of the proposed setup.

I. Introduction

Thermoacoustic instabilities have been studied extensively in the past and appear in a variety of natural and artificial phenomena. The simplest example of a thermoacoustically unstable system is a sealed duct where an external temperature differential is imposed on the walls. Under the right conditions, this system is capable of spontaneously generating acoustic power from the imposed energy source. Wave-induced, quasi-isentropic compressions and dilatations work against the background temperature gradient, spontaneously converting heat into acoustic power and are, thus, self-amplifying. This is the principle underlying traditional thermo-acoustic energy conversion, where gaseous fluids in the ideal gas regime (e.g. air or helium) are typically employed.

Fluids other than ideal gases can be theoretically selected as working substances. Practically, a liquid has been employed in Malone-type Stirling-like engines, and liquid sodium has been used successfully. To the best authors’ knowledge, the use of a supercritical fluid as a working substance for a thermoacoustically unstable duct has never been considered before.

We here present a theoretical investigation of a thermoacoustically unstable one-dimensional resonator with supercritical CO$_2$ as the working fluid. We show that this choice can significantly increase the acoustic power generation because of the strong variations of the thermodynamic base state properties, in particular the large drop in density for small temperature variations.

We perform linear stability calculations of a minimal unit resonator composed by a thermoacoustic stack bounded on the hot side by a cavity containing CO$_2$ in pseudo-gas phase and on the cold side by a resonator

*PhD student, Department of Mechanical Engineering, AIAA Member, email: migliom@purdue.edu
†PhD student, Department of Mechanical Engineering, AIAA Member, email: gupta288@purdue.edu
‡Assistant Professor, Department of Mechanical Engineering, AIAA Member, email: scalo@purdue.edu
II. Governing equations and acoustic energy budgets

II.A. Governing equations

The conservation of mass, momentum and total energy, can be written as

\[
\frac{D\rho}{Dt} = -\rho \frac{\partial u_j}{\partial x_j}, \quad \frac{D\rho u_j}{Dt} = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j}, \quad \frac{DE}{Dt} = -\frac{\partial p u_j}{\partial x_j} + \frac{\partial u_i \tau_{ij}}{\partial x_j} - \frac{\partial q_j}{\partial x_j},
\]

(1)

where \( t \) is time, \( x_j \) and \( u_j \) \((j = 1, 2, 3)\) are the components of position and velocity, \( \rho \) and \( p \) are the density and pressure, \( E = e + u_j u_j/2 \) is the specific total energy, sum of specific internal energy and kinetic energy, \( D/Dt \) is the material derivative, and no source terms are considered. The Newtonian viscous stress tensor \( \tau_{ij} \) and the Fourier heat flux are

\[
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial u_k}{\partial x_k}, \quad q_j = -k \frac{\partial T}{\partial x_j},
\]

(2)

where \( \delta_{ij} \) is the Kronecker delta, \( \mu \) is the dynamic viscosity, the bulk viscosity \( \lambda = \zeta - 2\mu/3 \) defines the second viscosity \( \zeta \), here considered \( \zeta = 0 \) (Stokes’s hypothesis), \( k \) is thermal conductivity and \( T \) is temperature.

From Eq. (1) it is possible to derive (see appendix A) the frequency-domain equations for the complex volume flow rate, \( \hat{U} \) (defined in Eq. (31) in appendix A) and pressure, \( \hat{p} \):

\[
\sigma \hat{U} = -\frac{A}{\rho_0} \frac{d\hat{p}}{dx} (1 - f_v), \quad \sigma \hat{p} = \frac{\rho_0 a_0^2/A}{1 + (\gamma - 1)/f_k} \left( \Theta \Phi_p - \frac{d}{dx} \right) \hat{U}.
\]

(3)

where \( \sigma = \alpha + i \omega \) is the eigenvalue of the system with growth rate \( \alpha \) and pulsation \( \omega = 2\pi f \), \( f \) being the frequency of oscillation. The thermoviscous functions \( (f_v, f_k) \) are defined by Eqs. (43) and (51) in appendix A. Base state quantities are denoted with a subscript 0, \( a_0 \) is the isentropic speed of sound, \( \nu_0 = \mu_0/\rho_0 \) is the kinematic viscosity and \( \Gamma_0 = c_p_0 \rho_0/\rho_0 \) is the Prandtl number, where \( c_p_0 = \gamma_0 c_p_0 \) is the specific isobaric heat capacity. Furthermore, \( A \) is the cross-sectional area of the duct such that \( A = h \cdot 1 \) for \( m = 0 \) and \( A = \pi r^2 \) and,

\[
\Theta = -\frac{1}{\rho_0} \frac{d\rho_0}{dx} = \frac{d\ln v_0}{dx},
\]

(4)

where \( v_0 = 1/\rho_0 \) is the specific volume, and

\[
\Phi_p = \frac{1}{1 - \frac{f_k}{1 - f_v}}.
\]

(5)

The base pressure \( P_0 \) is uniform and the variable \( \Theta = \alpha_{p_0} dT_0/dx \).

II.B. Real fluid effects on thermoacoustic energy production

The acoustic energy conservation equation for a quasi-one-dimensional system can be rigorously derived as done by Gupta et al.\textsuperscript{2} yielding,

\[
\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{I}}{\partial x} = \mathcal{P} - \mathcal{D},
\]

(6)

where the one-dimensional acoustic energy density, the instantaneous acoustic flux \( \mathcal{I} \), and the net energy production \( \mathcal{P} - \mathcal{D} \) are given by

\[
\mathcal{E} = \frac{1}{2} \frac{p^2}{\rho_0 a_0^2} + \frac{1}{2} \frac{\rho_0}{A} \left( \frac{U'}{A} \right)^2, \quad \mathcal{I} = \frac{p'U'}{A}, \quad \mathcal{P} - \mathcal{D} = \frac{p'q'}{A \rho_0 c_p_0 / \alpha_{p_0}} + \frac{\tau_w U'}{A^2},
\]

(7)
where $\tau'_w$ and $q'$ are the fluctuating wall shear and heat transfer defined by Eq. (33).

Introducing the cycle-average operator

$$\overline{(.)} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} (\cdot) dt,$$

and averaging Eq. (6) over one acoustic cycle and integrating axially over the domain, $L$, yields

$$\frac{d}{dt} \int_L \mathcal{T} dx = \mathcal{R} = \int_L \left( \mathcal{P} - \mathcal{D} \right) dx,$$

where $\mathcal{R}$ is the Rayleigh index, which allows to identify the onset of an instability via the criterion $\mathcal{R} > 0$. 

Utilizing the frequency domain expression for the wall-heat flux $\hat{q}$ (Eq. (50)), the wall-shear stress $\hat{\tau}_w$ (second of Eq. (41)), and Eq. (42), allows to write

$$P = \chi^2 A h T_P < \hat{p} \hat{U} \chi >,$$

and

$$D = q D \frac{\tau}{2} \left( \frac{1}{1 + \frac{1}{Pr_0}} \right)^2 \frac{\tau}{2} \left( \frac{1}{1 + \frac{1}{Pr_0}} \right),$$

represent losses due to wall heat and wall shear, respectively. Notice that all the expressions above are identical to the ones derived in Gupta et al., with the only difference that the dependency from $1 - Pr_0$ is absorbed by the terms in Eq. (12). It can be shown that the weights in Eq. (10) always contribute positively overall to the thermoacoustic production regardless of the sign of $1 - Pr_0$.

![Figure 1. Prefactors $\Phi^T_P$ and $\Phi^S_P$ in Eq. (12) for rectilinear geometry versus dimensionless pore size at different Prandtl numbers.](image)

All the thermoviscous functions and the quantities in Eqs. (12) and (13) vary with the ratio of regenerator half-width $h/2$ to the Stokes boundary layer thickness $\delta_v$ (figures 1, 2, and 3), where the viscous and thermal boundary layers $\delta_v$, $\delta_k$ are defined by

$$\delta_v^2 = \frac{2\nu_0}{\omega}, \quad \delta_k^2 = \frac{2k_0}{\omega \rho_0 c_p}, \quad \delta_v = \sqrt{Pr_0} \delta_k.$$

In the case of $Pr_0 = 1$, $f_k = f_v$, and for a rectilinear geometry

$$\lim_{Pr_0 \to 1} \Phi_P = \frac{1}{2} \left( -1 + \frac{\nu_0 f_v^2}{1 - f_v} \right),$$
If instead $\Pr_0 = 0$, $f_k = 1$, and

$$\lim_{\Pr_0 \to 0} \Phi_P = 1,$$

indicating that $\Phi_P^T = 1$ and $-\Phi_P^S = 0$, so that the production due to standing waves is null.

Having a Prandtl number close to 0 entails high thermoacoustic production over a wide range of $h/(2 \delta_v)$, but for $\Pr_0 > 1$, $\Phi_P^T$ tends to $\Pr_0 - 1$ for $h/(2 \delta_v) = 0$, decreasing much faster than the $\Pr_0 = 0$ curve for increasing $h/(2 \delta_v)$.

With the normal mode assumption,

$$p' = \Re(\hat{p}(x)e^{(\alpha + i \omega) t}), \quad U' = \Re(\hat{U}(x)e^{(\alpha + i \omega) t}),$$

and assuming that in the (fast) time scale of $\omega$ we can neglect the (slow) time scale of $\alpha$, we have

$$\mathcal{E} = \frac{e^{2\alpha t}}{4} \left( \frac{1}{\rho_0 \alpha_0^2} |\hat{p}(x)|^2 + \frac{\rho_0}{A^2} |\hat{U}(x)|^2 \right)$$

and therefore the acoustic energy budget of Eq. (10) can be closed as

$$\alpha = \frac{2 \mathcal{R}}{\mathcal{E}_{tot}},$$

where

$$\mathcal{E}_{tot} = \int_L \mathcal{E}.$$
III. Computational setup

III.A. Fluid model

Higher thermoacoustic production is achieved when the variable $\Theta$ is increased (Eq. (10)). This consideration sets the guidelines for the selection of the base state for a fixed resonator geometry.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>$T_{cr}$ (K)</th>
<th>$p_{cr}$ (MPa)</th>
<th>$\rho_{cr}$ (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO$_2$</td>
<td>304.1282</td>
<td>7.3773</td>
<td>467.6</td>
</tr>
</tbody>
</table>

Table 1. Carbon dioxide critical parameters.

Carbon dioxide in supercritical conditions is chosen as working fluid (table 1, figure 4a). From heavy pseudo-liquid (PL), the density dramatically falls through a pseudo-boiling (PB) process, after which the fluid transitions to a pseudo-gaseous (PG) state, reaching eventually, a near-ideal-gas state (IG)$^6$ (figure 4b). We here define the PB state as the region in the immediate vicinity of the spike of $\Theta$ or of $d(\ln \rho_0^{-1})/dT$. In effects, $d(\ln \rho_0^{-1})/dT$ is a $\Theta$ per unit $dx/dT$.

![Phase diagram for carbon dioxide showing flooded contours of reduced density $\rho/\rho_{cr}$](image1)

![Density variation versus temperature for CO$_2$ at $p = 1.1p_{cr}$](image2)

III.B. Computational setup

The computational setup (figure 5) is a two-dimensional duct composed by a hot cavity, indicated by the subscript $cav$, the thermoacoustic stack, indicated by the subscript $stk$, and a resonator. The whole domain is enclosed by two adiabatic walls at $x = 0$ and $x = L$, respectively. Fluid at PL conditions ($T_{cold}$) and PG conditions ($T_{hot}$) bound the stack, kept in pseudo-boiling conditions. This setup represents the minimal unit of a thermoacoustic device.

![Computational setup showing regions of pseudo-gas (PG) and pseudo-liquid (PL) with fixed geometrical parameters](image3)

Along the walls of the stack ($\ell_{cav} < x < \ell_{cav} + \ell_{stk}$), isothermal boundary conditions impose the following
temperature distribution

\[
T_0 = \begin{cases} 
\frac{T_{\text{hot}} + T_{\text{cold}}}{2} & 0 \leq x \leq \ell_{\text{cav}} \\
\frac{T_{\text{hot}} - T_{\text{cold}} \text{erf}[\beta(x - \ell_{\text{cav}} - \ell_{\text{stk}}/2)]}{T_{\text{cold}}} & \ell_{\text{cav}} < x < \ell_{\text{cav}} + \ell_{\text{stk}} \\
T_{\text{cold}} & \ell_{\text{cav}} + \ell_{\text{stk}} \leq x \leq L
\end{cases}
\]  

(21)

where \( \beta = 75 \) and the variable \( T_{\text{cold}} \) is varied accordingly to table 2, defining 6 computational cases (figure 6a).

<table>
<thead>
<tr>
<th>symbol</th>
<th>( T_{\text{cold}}/T_{\text{cr}} )</th>
<th>( \rho_{\text{cold}} ) (kg/m(^3))</th>
<th>( \rho_{\text{hot}} ) (kg/m(^3))</th>
<th>( T_{\text{cold}} ) (K)</th>
<th>( T_{\text{hot}} ) (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>▼</td>
<td>0.65</td>
<td>1314.8</td>
<td>258.5</td>
<td>197.7</td>
<td>317.7</td>
</tr>
<tr>
<td>▲</td>
<td>0.70</td>
<td>1264.1</td>
<td>201.7</td>
<td>212.9</td>
<td>332.9</td>
</tr>
<tr>
<td>⋄</td>
<td>0.75</td>
<td>1206.3</td>
<td>173.7</td>
<td>228.1</td>
<td>348.1</td>
</tr>
<tr>
<td>●</td>
<td>0.80</td>
<td>1139.6</td>
<td>155.3</td>
<td>243.3</td>
<td>363.3</td>
</tr>
<tr>
<td>■</td>
<td>0.85</td>
<td>1060.8</td>
<td>141.8</td>
<td>258.5</td>
<td>378.5</td>
</tr>
<tr>
<td>△</td>
<td>0.90</td>
<td>964.1</td>
<td>131.3</td>
<td>273.7</td>
<td>393.7</td>
</tr>
</tbody>
</table>

Table 2. Temperature settings explored in the present study for the thermoacoustic resonator (figure 5).

The base temperature and pressure set all the other base state variables through the fluid model. The Peng-Robinson\(^7\) equation of state (EoS) and Chung’s model\(^1,8\) are chosen as real-fluid model due to their simplicity and accuracy for the range of variables of this study.\(^5\)

![Figure 6](image)

Figure 6. Base temperature, (a); density, (b); \( \Theta \) (c); and position of the pseudo-boiling region inside the thermoacoustic stack (d). Case-specific symbols (table 2) are placed at locations corresponding to pseudo-boiling conditions. The dashed lines in (c) represent \( \Theta \) that would be obtained if an ideal gases EoS is implemented (for the same conditions of table 2).

Some of the base state variables are of interest. The base density \( \rho_0 \) (figure 6b) rapidly drops in the PB region giving rise to a high spike in \( \Theta \) inside the thermoacoustic stack (figure 6c). If an ideal gas were to be employed at the same conditions, \( \Theta \) would decrease by two orders of magnitude.
The base Prandtl number (figure 7a), the variable $h/(2\delta_r)$ (figure 7b), and the gradient of the base impedance (figure 7d) all spike at PB conditions. The latter, in particular, modifies acoustic perturbations akin to a change in geometry, therefore changing the spatial derivatives of the eigenmodes.

![Figure 7. Base state quantities inside the thermoacoustic stack: (a) Prandtl number; (b) ratio of regenerator half-width $h/2$ to the Stokes boundary layer thickness $\delta_r$; (c) base impedance $Z_0 = \rho_0 a_0$; (d) base impedance gradient in x.](image)

### IV. Results

Eqs (42) and (52) constitute a generalized eigenvalue problem that we solve, with the Arnoldi shift-invert algorithm, obtaining frequencies and growth rates (figure 8), and eigenmodes of the thermoacoustic resonator (figure 9).

![Figure 8. Growth rates and frequencies for the computational cases.](image)

The growth rate and frequency of the system are extremely sensitive to the choice of the working fluid inside the resonator. Growth rates show a non-monotonic trend with increasing ratio $T_{cold}/T_{cr}$ reaching maximum at $T_{cold}/T_{cr} \approx 0.75$. Moreover, the given geometry exhibits linear stability for ideal gas as the working fluid within the same temperature range (not shown). The pressure eigenmode exhibits a node close to the thermoacoustic stack and an anti-node on the right most end due to the high free-field acoustic
impedance of the cold (and heavy) carbon dioxide (figure 7c). Moreover, the flow rate eigenmode also
 corresponds to a shape similar to the first harmonic of the resonator. Inside the thermoacoustic stack,
 discontinuity in the derivative of eigenmodes occurs at $x = x_{PB}$ due to abrupt peak in the spatial derivative
 of the base acoustic impedance. The highly dense pseudo-liquid on the right of the pseudo-boiling region has
 high base impedance resulting in an increase in the pressure amplitude at the resonator (right end) compared
 to the hot cavity (left end). Usually, for thermoacoustic resonators in standing wave configurations, the
 pressure amplitude is higher inside the hot cavity compared to the resonator.

![Graphs showing amplitude of pressure and flow rate eigenmodes](image)

**Figure 9.** Amplitude of pressure (left) and flow rate (right) eigenmodes for the conditions of table 2. The vertical lines
represent $x = \ell_{cav}$, beginning of the regenerator, and $x = \ell_{cav} + \ell_{stk}$, end of the regenerator.

Throughout the thermoacoustic stack, the normalized amplitude of the impedance (figure 10a) changes
by an order of magnitude because of the axial variation of base impedance $\rho_0 a_0$, while the phase (figure 10b)
and amplitude of the dimensional impedance (figure 11a) do not change significantly. The positive spatial
gradient of the acoustic flux around the pseudo-boiling region (figure 11b) inside the thermoacoustic stack
indicates thermoacoustic production, as expected for the linearly unstable configuration.

![Graphs showing acoustic impedance](image)

**Figure 10.** Acoustic impedance $Z = \hat{p}/\hat{U}$: (a) normalized amplitude; (b) phase angle.
Acknowledgments

This work has been performed with the support of the Frederick N. Andrews and Rolls-Royce Doctoral Fellowships at Purdue University. The authors thank the fruitful discussions with Mr. Pat Sweeney (Rolls-Royce) and Prof. Stephen D. Heister (Purdue) for having inspired this work. The authors thank Jean-Pierre Hickey for useful discussions about supercritical fluids. The computing resources were provided by the Rosen Center for Advanced Computing (RCAC) at Purdue University and Information Technology at Purdue (ITaP).

References


A. Derivation of the harmonic equations

A.A. Other forms of energy conservation

The last equation in (1) can be rewritten as the evolution equation for the specific internal energy, $e$, or specific enthalpy, $h = e + p/\rho$, or specific entropy, $s$,

$$
\rho \frac{De}{Dt} = -p \frac{\partial u_j}{\partial x_j} + \rho T \frac{Ds}{Dt}, \quad \rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \rho T \frac{Ds}{Dt}, \quad \rho T \frac{Ds}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_j},
$$

(22)

thanks to the Gibbs relations.

The equation of state $p = p(\rho, T)$ can be differentiated to obtain

$$
\kappa_T dp = v d\rho + \alpha_p dT,
$$

(23)
where \( v = 1/\rho \) is the specific volume and

\[
\kappa_T = \left. \frac{1}{v} \frac{\partial v}{\partial p} \right|_T = \gamma \rho a^2, \quad \alpha_p = \left. \frac{1}{v} \frac{\partial v}{\partial T} \right|_p = \kappa_T \left. \frac{\partial p}{\partial T} \right|_v,
\]

are the isothermal compressibility and the isobaric thermal expansion coefficient.

The Gibbs relation \( dh = c_p dT + v(1 - \alpha_v T) dp \), after using Eq. (23) and the relation \( c_p - c_v = \alpha_T^2 T v / \kappa_T \)
where \( c_p = \gamma c_v \) is the isobaric specific thermal capacity, allows to rewrite the enthalpy evolution equation as the pressure or temperature evolution equations,

\[
\frac{D p}{Dt} = -\rho a^2 \frac{\partial u}{\partial x} + \frac{\rho a^2 \alpha_p T Ds}{c_p} \quad \rho c_v \frac{DT}{Dt} = -\rho c_v (\gamma - 1) \frac{\partial u}{\partial x} + \rho T Ds.
\]

A.B. Linearized 2D equations

The mass and momentum equations in cartesian \((m = 0)\) or cylindrical \((m = 1)\) coordinates are, with \( x \) \((j = 1)\) corresponding to the axial direction and \( r \) \((j = 2)\) to the radial direction,

\[
\frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j} = -\rho \left( \frac{\partial u}{\partial x} + \frac{1}{r^m} \frac{\partial(r^m u)}{\partial r} \right), \quad \rho \left( \frac{\partial u}{\partial t} + u_j \frac{\partial u}{\partial x_j} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial r} \left( r^m \frac{\partial u}{\partial r} \right),
\]

where we neglected axial viscous stresses, disregarded the momentum equation in the \( r \) direction \((\partial p/\partial r = 0)\), and assumed that \( v = v(r,t) \) and \( \mu = \mu(x,t) \). The pressure evolution equation becomes

\[
\frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j} = a^2 \left( \frac{\partial u}{\partial t} + u_j \frac{\partial u}{\partial x_j} \right) + \frac{\alpha_p a^2}{c_p} \left( \tau_{ij} \frac{\partial u_{ij}}{\partial x_j} + \frac{k}{r^m} \frac{\partial}{\partial r} \left( r^m \frac{\partial T}{\partial r} \right) \right),
\]

where we assumed that \( k = k(x,t) \) and neglected axial heat conduction.

A.C. Linearized 2D equations

A first order expansion is assumed,

\[
\rho = \rho_0(x) + \rho'(x, y, t), \quad T = T_0(x) + T'(x, y, t), \quad p = p_0 + p'(x, y, t), \quad u = u'(x, y, t), \quad v = v'(y, t),
\]

where a generic variable is composed by a base quantity, at most varying axially, and a fluctuation that is time and space dependent, and no mean flow is considered. Neglecting nonlinear terms, Eq. (26) becomes

\[
\frac{\partial p'}{\partial t} + u_j \frac{\partial p_0}{\partial x_j} = -\rho_0 \left( \frac{\partial u'}{\partial x} + \frac{1}{r^m} \frac{\partial(r^m u')}{\partial r} \right), \quad \rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{\mu_0}{r^m} \frac{\partial}{\partial r} \left( r^m \frac{\partial u'}{\partial r} \right).
\]

and Eq. (27) reads

\[
\frac{\partial p'}{\partial t} = a_0^2 \left( \frac{\partial p'}{\partial t} + u_j \frac{\partial p_0}{\partial x_j} \right) + \frac{\gamma_0 - 1}{\alpha_p T_0} \frac{\partial}{\partial r} \left( r^m \frac{\partial T'}{\partial r} \right),
\]

where we used the relation \( a^2 T a_0^2/\gamma = \gamma - 1 \).

A.D. Quasi-one-dimensional linearized equations

We introduce

\[
U' = \int_{-h/2}^{h/2} u'(x, r, t)(2\pi r)^m dr, \quad A = \int_{-h/2}^{h/2} (2\pi r)^m dr,
\]

where \( U' \) is the mass flow rate fluctuations. Integrating Eqs. (29) and (30) over the cross section, we obtain

\[
\rho_0 \frac{\partial U'}{\partial t} = -A \frac{\partial p'}{\partial x} + \tau_w', \quad A \frac{\partial p'}{\partial t} = -\rho_0 a_0^2 \frac{\partial U'}{\partial x} + \frac{\gamma_0 - 1}{\alpha_p T_0} q',
\]

where we accounted for impenetrable boundary conditions in \( y \), and

\[
\tau_w' = 2\rho_0 \left( \frac{\pi h}{2} \right)^m \frac{\partial u'}{\partial r} \qquad q' = 2k_0 \left( \frac{\pi h}{2} \right)^m \frac{\partial T'}{\partial r}.
\]

are the wall shear stress and the wall heat flux assuming vertical or angular symmetry of the fluctuations.
A.E. Harmonic form of the linearized 2D equations

In the following we will assume an harmonic dependence of the perturbations (normal mode assumption),

\[
\rho' = \hat{\rho}(x,r)e^{\sigma t}, \quad T' = \hat{T}(x,r)e^{\sigma t}, \quad p' = \hat{p}(x)e^{\sigma t}, \quad u' = \hat{u}(x,r)e^{\sigma t}, \quad v' = \hat{v}(r)e^{\sigma t}.
\] (34)

Thus, Eq (29) becomes

\[
\sigma \hat{\rho} + \frac{\partial \hat{\rho}}{\partial x} + \frac{d \rho_0}{dx} + \frac{\rho_0}{r^m} \frac{\partial \hat{\rho}}{\partial r} \hat{v} = 0, \quad \sigma \hat{u} + \frac{1}{\rho_0} \frac{d \hat{p}}{dx} = \frac{\nu_0}{r^m} \frac{\partial}{\partial r} \left( r^m \frac{\partial \hat{u}}{\partial r} \right),
\] (35)

and Eq. (30) becomes

\[
\sigma \left( \hat{p} - \frac{1}{\alpha_0^2} \hat{p} \right) + \hat{u} \frac{d \rho_0}{dx} = \frac{\nu_0}{\rho_0} \frac{1}{r^m} \frac{\partial}{\partial r} \left( r^m \frac{\partial \hat{p}}{\partial r} \right),
\] (36)

that are the harmonic mass, momentum, and energy equations. Notice that despite not having made any assumption on the fluid, Eq. (35) and (36) coincide with the equations proposed by Rott,10 who used the ideal gas EoS. Eq. (32) therefore becomes

\[
\rho_0 \sigma \hat{U} = -A \frac{d \hat{p}}{dx} + \hat{\tau}_w, \quad A \sigma \hat{p} = -\rho_0 \alpha_0^2 \hat{U} + \frac{\gamma_0 - 1}{\alpha_0 \bar{T}_0} \hat{q},
\] (37)

A.F. Analytical solution in the radial direction

In this section we solve in the radial direction the momentum and energy equations considering the flow as purely oscillatory, i.e. \( \alpha \ll \omega \) and therefore \( \sigma \approx \omega \). This is generally a fair assumption since in one acoustic cycle no waves grow or decreases dramatically.

We employ a well known procedure,2,3,10 rederiving Rott’s theory for a generic fluid,11 starting from the coordinate transformation

\[
\xi = i \eta, \quad \eta = \frac{\sqrt{\omega \rho_0}}{\nu_0}, \quad \hat{u}_* = -i \omega \rho_0 \hat{u} \left( \frac{d \hat{p}}{dx} \right)^{-1} - 1,
\] (38)

where notice that \( \eta = r \sqrt{2} / \delta_r = -(i - 1) r / \delta_r \). Recalling that pressure modes do not vary radially, Eq. (38) allows to rewrite the momentum equation as

\[
\xi^2 \frac{\partial^2 \hat{u}_*}{\partial \xi^2} + m \xi \frac{\partial \hat{u}_*}{\partial \xi} + \xi^2 \hat{u}_* = 0,
\] (39)

that, in the case of \( m = 1 \), is a Bessel’s differential equation of order 0. Eq. (39) has the general solution

\[
(m = 1) : \quad \hat{u}_*(\xi) = -J_0(\xi) / J_0(\xi_b), \quad (m = 0) : \quad \hat{u}_*(\eta) = -\cosh(\eta) / \cosh(\eta_b)
\] (40)

where \( J_0 \) is the Bessel function of the first kind of order 0, the subscript \( b \) indicates evaluation at \( r = h / 2 \), i.e. \( \eta_b = \sqrt{\omega \rho_0 b / 2} \), and the no-slip boundary condition has been taken into account \( \hat{u}_*(\xi_b) = \hat{u}_*(\eta_b) = -1 \).

Therefore the solution of the momentum equation and the wall shear stress in frequency domain are

\[
\hat{u} = -\frac{1}{i \omega \rho_0} \frac{d \hat{p}}{dx} \left( \hat{u}_* + 1 \right), \quad \hat{\tau}_w = A \frac{d \hat{p}}{dx} f_\nu,
\] (41)

and the first of Eq. (37) is

\[
\sigma \hat{U} = - \frac{A}{\rho_0} \frac{d \hat{p}}{dx} (1 - f_\nu),
\] (42)

where

\[
(m = 1) : \quad f_\nu = \frac{2 J_1(\xi_b)}{\xi_b J_0(\xi_b)}, \quad (m = 0) : \quad f_\nu = \frac{\tanh(\eta_b)}{\eta_b}.
\] (43)

By incorporating the isothermal boundary conditions at the walls, Eq. (23) becomes

\[
\gamma_0 \frac{\hat{p}_w}{\rho_0 a_0^2} = \frac{\hat{p}_w}{\rho_0} + \alpha_0 \hat{\nu}_w \rightarrow \hat{p}_w = \frac{\gamma_0}{a_0^2} \hat{p}.
\] (44)
where we used the independency of $\hat{p}$ from $r$. Using the first two of Eq. (38), Eq. (36) becomes

$$
\frac{\partial^2}{\partial \xi^2}(\hat{\rho} - \hat{\rho}_w) + \frac{m}{\xi} \frac{\partial}{\partial \xi}(\hat{\rho} - \hat{\rho}_w) + Pr_0(\hat{\rho} - \hat{\rho}_w) = -Pr_0 \frac{\gamma_0 - 1}{a_0^2} \frac{d}{dx} \hat{p} - \frac{Pr_0}{i\omega} \frac{d\rho}{dx},
$$

(45)

that is the inhomogeneous analog of Eq. (39). To revert back to an homogeneous equation, we assume a solution of the form

$$
(\hat{\rho} - \hat{\rho}_w) = -\frac{\rho_0 \Theta}{i\omega} \frac{Pr_0}{1 - Pr_0} \hat{u}(\xi) + (\rho - \hat{\rho}_w),
$$

(46)

where we assumed $Pr_0 \neq 1$ everywhere axially, obtaining from Eq. (45) the equation

$$
\xi^2 \frac{\partial^2 \hat{\rho}_*}{\partial \xi^2} + m \xi \frac{\partial \hat{\rho}_*}{\partial \xi} + \xi^2 \hat{\rho}_* = 0, \quad \xi = \xi_0 \sqrt{Pr_0}, \quad \hat{\rho}_* = -(\hat{\rho} - \hat{\rho}_w)^2 \left(\frac{\gamma_0 - 1}{a_0^2} \hat{p} - \frac{\Theta}{(1 - Pr_0) \omega^2} \frac{d\hat{p}}{dx}\right)^{-1}
$$

(47)

that can be solved with a solution of the form of Eq. (40) replacing $\xi$ with $\bar{\xi}$, taking advantage of the wall boundary conditions. We can therefore write the perturbation density equation:

$$
(m = 1): \quad \hat{\rho} - \hat{\rho}_w = (\frac{\Theta}{(1 - Pr_0) \omega^2} \frac{d\hat{p}}{dx} - \frac{\gamma_0 - 1}{a_0^2} \hat{p})(1 - \frac{J_0(\xi_0 \sqrt{Pr_0})}{J_0(\xi_0 \sqrt{Pr_0})}) - \frac{\Theta \frac{Pr_0}{J_0(\xi_0 \sqrt{Pr_0})}}{\omega^2} \frac{d\hat{p}}{dx} \left(1 - \frac{J_0(\xi)}{J_0(\xi_0)}\right),
$$

(48)

$$
(m = 0): \quad \hat{\rho} - \hat{\rho}_w = (\frac{\Theta}{(1 - Pr_0) \omega^2} \frac{d\hat{p}}{dx} - \frac{\gamma_0 - 1}{a_0^2} \hat{p})(1 - \frac{\cosh(\eta \sqrt{Pr_0})}{\cosh(\eta \sqrt{Pr_0})}) - \frac{\Theta \frac{Pr_0}{\cosh(\eta \sqrt{Pr_0})}}{\omega^2} \frac{d\hat{p}}{dx} \left(1 - \frac{1}{\cosh(\eta)}\right).
$$

(49)

Here, as noticed before, the derivation of Rott, performed under with the assumption of ideal gases, still holds for a generic fluid, in the case of $Pr_0 \neq 1$.

We can now evaluate the wall heat flux in the frequency domain,

$$
\hat{q} = i\omega A \frac{\rho_0}{\alpha_{po}} \left(\frac{\Theta}{(1 - Pr_0) \omega^2} \frac{d\hat{p}}{dx} (f_k - f_v) - \frac{\gamma_0 - 1}{a_0^2} \hat{p}f_k\right),
$$

(50)

where notice that the term $1/\alpha_{po}$ reverts back to $T_0$ in the case of ideal gas EoS,$^2$ and where

$$
(m = 1): \quad f_k = \frac{2}{\xi_0 \sqrt{Pr_0}} \frac{J_1(\xi_0 \sqrt{Pr_0})}{J_0(\xi_0 \sqrt{Pr_0})}, \quad (m = 0): \quad f_k = \frac{\tanh(\eta \sqrt{Pr_0})}{\eta \sqrt{Pr_0}}.
$$

(51)

Eqs (43) and (51) define the thermoviscous functions.

Finally, the pressure equation in harmonic form can be written as

$$
\sigma \hat{p} = \frac{\rho_0 a_0^2 / A}{1 + (\gamma_0 - 1) f_k} \left(\frac{\Theta (f_k - f_v)}{(1 - f_v)(1 - Pr_0)} - \frac{d}{dx}\right) \hat{U}.
$$

(52)