Semester-end summary Fall 2019

We did 4 things in this class

0. translate words into math
1. build models.
2. compute probabilities (mostly mechanical)
3. infer
4. summary stats (mostly mechanical)

2. compute probs.

axioms $P(A^c) = 1 - P(A)$, etc.

Relies on identifying the event of interest

$$P(a < X \le b) = F_X(b) - F_X(a)$$

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$P((X,Y) \in R) = \iint_{(x,y) \in R} f_{XY}(x,y)\,dx\,dy$$

$$P(a < X \le b,\ c < Y \le d) = F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c)$$

double integration

(continuous or discrete) (or mixed)

1. Building models

a. Conditional probability

$$P(A) = \sum_i P(A \mid B_i)\, P(B_i)$$

$$f_X(x) = \sum_i f_X(x \mid B_i)\, P(B_i)$$

$$f_X(x) = \int f_X(x \mid y)\, f_Y(y)\, dy$$

Theorems of Total probability

b. Independence

$$P(A \cap B) = P(A)P(B)$$

$$f_{XY}(x,y) = f_X(x) f_Y(y) \quad \text{for all } x, y$$

c. Common pmfs (Bernoulli, binomial, geometric, ...)

d. Common pdfs (Gaussian, exponential)

e. $Y = g(X)$ derived RV

$$Z = g(X, Y)$$

not covered this semester

f. using MGF for $Z = X + Y$, where X and Y are independent

g. Strict and Wide sense stationarity

i) $m_X(t)$ is constant for all time t

ii) $R_X(t, t+\tau)$ is a function of τ, but <u>not</u> of t

h. If $H(f)$ is transfer function of LTI system, then

$$S_Y(f) = |H(f)|^2 S_X(f)$$

3. Inference

a. Bayes $P(B_i|A) = \dfrac{P(A|B_i)\,P(B_i)}{P(A)}$

b. $f_X(x|C) = \begin{cases} \dfrac{f_X(x)}{P(C)} & x \in C \\ 0 & \text{else} \end{cases}$

when C is an event that includes X.

c. $f_Y(y|x) = \dfrac{f_X(x/y)\,f_Y(y)}{f_X(x)}$

Applications: X depends on a parameter Y. Unknown Y is modeled probabilistically

d. Hypothesis Testing

Decide H_1 or H_0 depending on observation

e. Parameter estimation

Minimum MSE estimator

$\hat{x} = E(X|y)$

Linear MMSE estimator

$\hat{x} = ay + b$

not covered this semester

MAP estimator

$\hat{x} = \underset{x}{\arg\max}\; f_X(x|y)$

ML estimator

$\hat{x} = \underset{x}{\arg\max}\; f_Y(y|x)$

4. Compute summary statistics

a. Mean, variance, moments

$E(X)$, $Var(X)$, $E(X^n)$

not covered this semester

b. Moment generating function

$$M_X(s) = E(e^{sX}) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

c. Conditional mean and conditional variance

$$E(X|A) = \int_{-\infty}^{\infty} x\, f_X(x|A)\,dx$$

Theorem total expectation

$$E(X) = \sum_{i=1}^{n} E(X|B_i)\,P(B_i)$$

if B_i's form a partition

$$E(X) = E(E(X|Y))$$

Iterated expectations

d. Second moments for pairs of RVs

Correlation $r_{XY} = E(XY)$

covariance $c_{XY} = E(XY) - E(X)E(Y)$

correlation coefficient $\rho_{XY} = \dfrac{c_{XY}}{\sigma_X \sigma_Y}$

e. Law of Large Numbers and Central Limit Theorem

$M_n = \frac{1}{n}\sum X_i$; $E(M_n) = E(X)$; $Var(M_n) = \dfrac{Var(X)}{n}$

As n increases, PDF of M_n becomes narrower and CDF of M_n becomes well-approximated by Gaussian

f. Second moments for Random Processes

Auto correlation $\quad R_X(t_1, t_2) = E(X(t_1)X(t_2))$

Auto covariance $\quad C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)$

Cross correlation $\quad R_{XY}(t_1, t_2) = E(X(t_1)Y(t_2))$

Cross covariance $\quad C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2)$

Power spectral Density

$S_X(f)$ = Fourier Transform of $R_X(\tau)$