Power Spectral Density function (PSD) (Chapter 10.1)

Definition: If $X(t)$ is a WSS RP w/ $R_X(\tau)$ then the Power Spectral density of $X(t)$ is

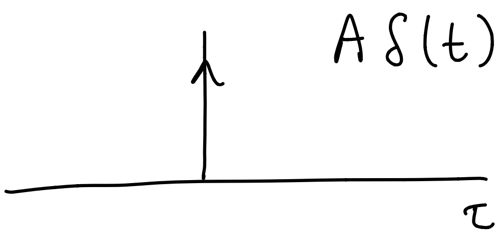
$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

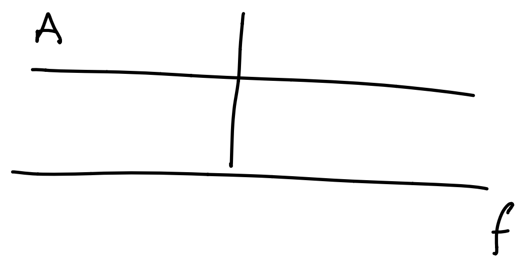
the Fourier Transform of the autocorrelation function

A measure of the average distribution of signal power as a function of frequency.
It is only defined for WSS RPs

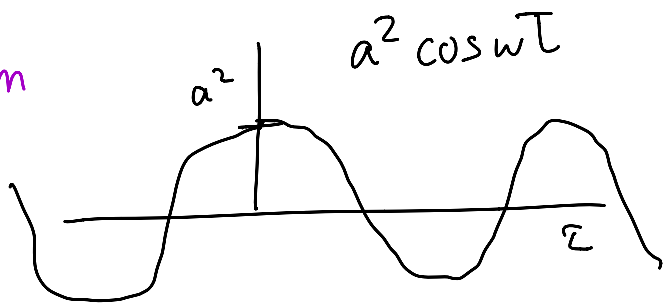
Examples $\quad R_X(\tau) \quad\quad S_X(f)$

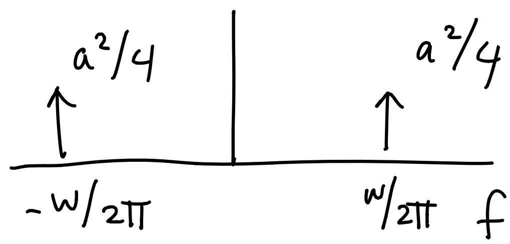
white noise



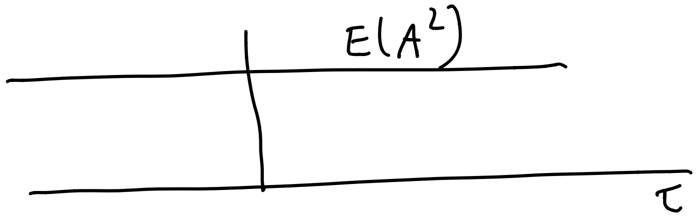


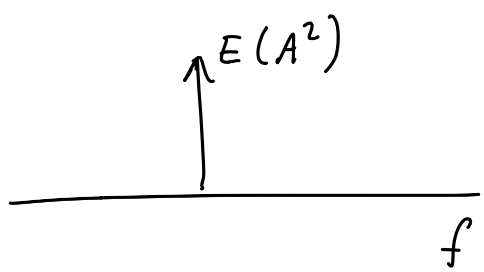
cosine w/ random phase





a constant RP $X(t)=A$





Intuition and power spectral density function

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

① PSD is just auto-correlation in another form, so it's a different way to represent the same thing

② Sometimes we can get more insight from one form than another. But intuition can be obtained from either

$R_X(\tau) = E(X(t)\,X(t+\tau))$

τ represents how far apart the samples we're considering are.

ⓐ If there is still a high correlation when τ is large, then $X(t)$ must not vary quickly. If $X(t)$ doesn't vary quickly, it must be comprised of mostly <u>low frequencies</u>.

ⓑ If a low correlation when τ small, then $X(t)$ typically varies quickly
$\Rightarrow$ comprised mostly of high frequencies

Properties of PSD.

① $S_X(f)$ is a real function

(because $R_X(\tau)$ is even symmetric)

② $$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f t}\, df$$

③ Cannot compute $S_X(f)$ unless $X(t)$ is WSS

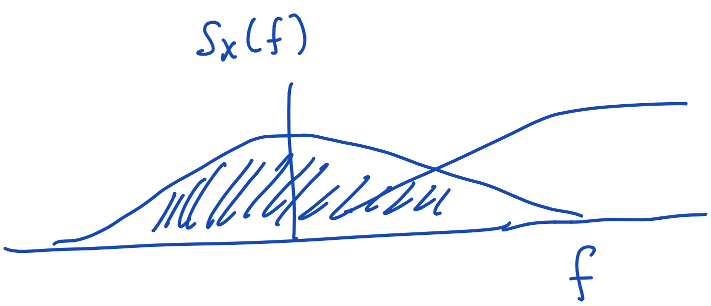
④ $S_X(f) \geq 0$ always

⑤ Average power

$$E(X^2(t)) = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df$$

computed by integrating the PSD over all frequencies

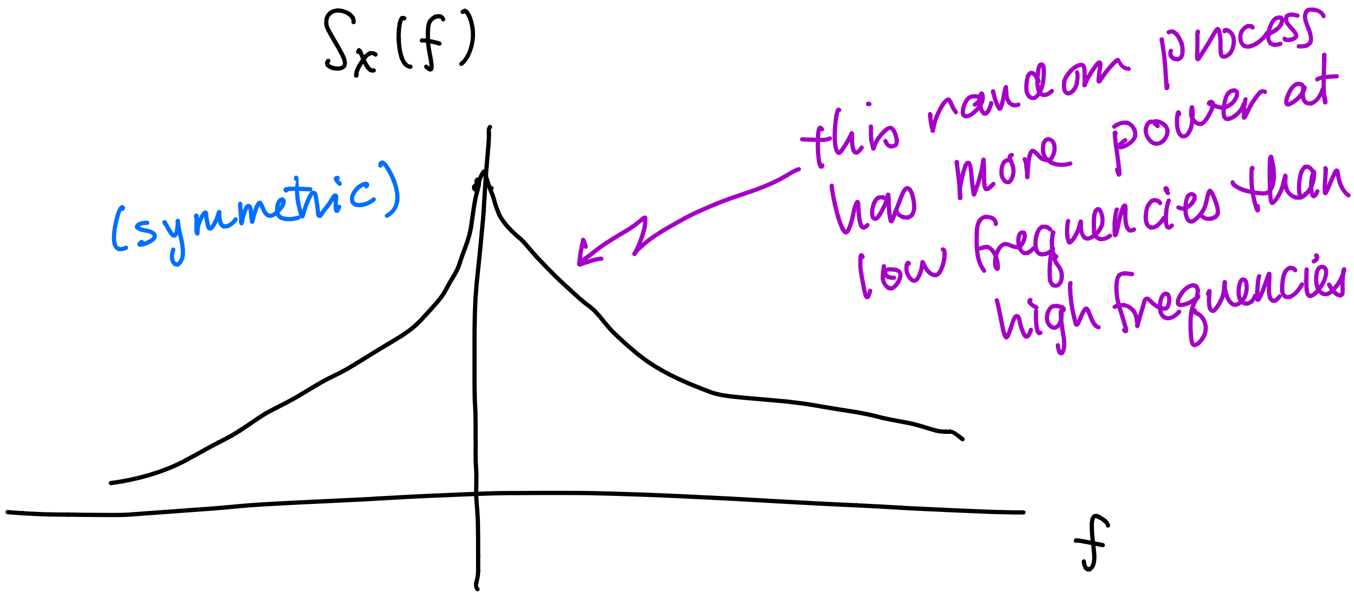
(which is consistent with the notion of a "power spectral density"

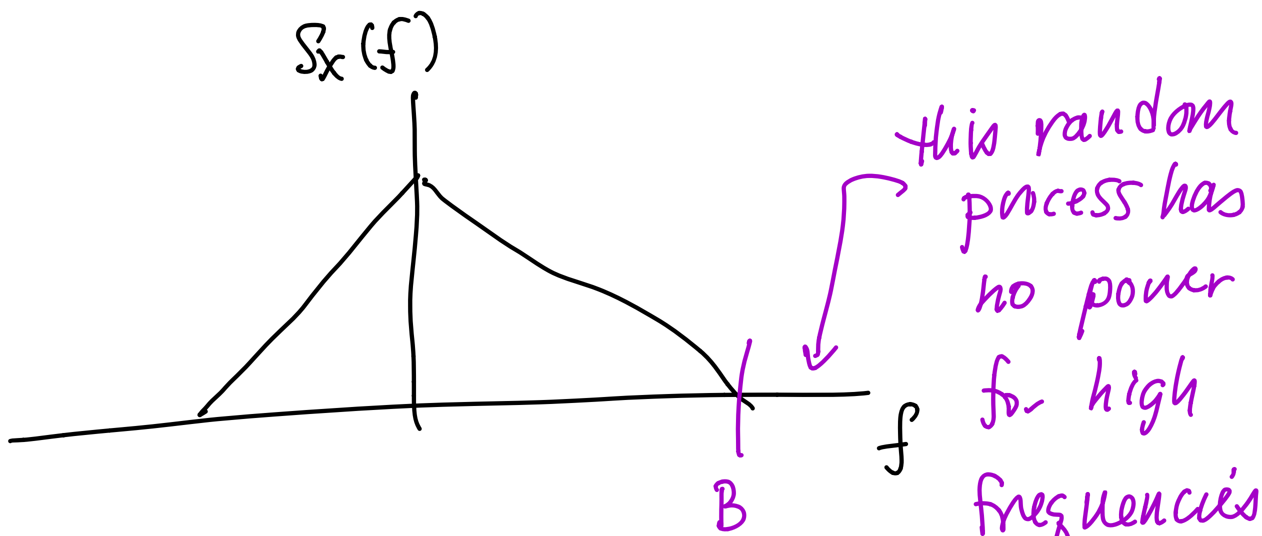


area under curve is average power

$$\int_{-\infty}^{\infty} S_X(f)\, df$$

PSD is also a measure of the average distribution of signal power as a function of frequency





$\Rightarrow$ band limited RP

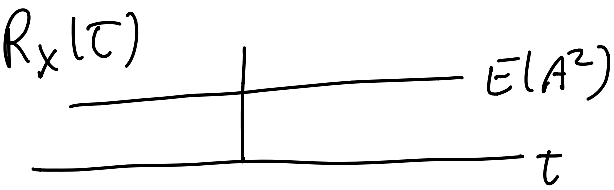
Examples of PSD

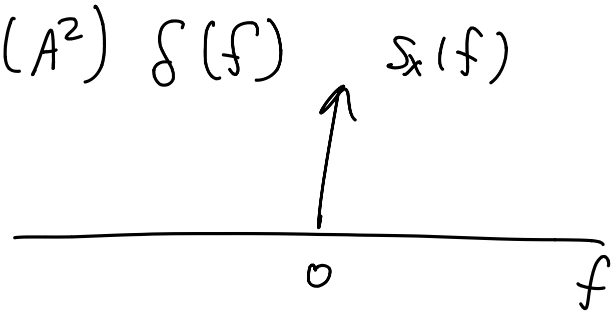
① a constant RP $\quad X(t) = A \quad$ where A is a RV

- This is WSS because $E(X(t)) = \text{constant}$
  and $E(X(t_1)X(t_2)) =$ a function of $(t_2 - t_1)$

$$R_X(\tau) = E(X(t)X(t+\tau)) = E(A^2)$$

$$S_X(f) = \mathcal{F}(R_X(\tau)) = E(A^2)\,\delta(f)$$





② Cosine w/ random phase

Θ uniform on $[0, 2\pi)$

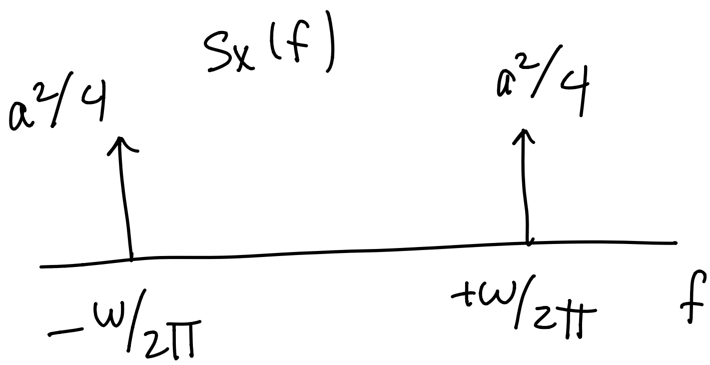
$X(t) = a\cos(\omega t + \Theta) \qquad$ constant amplitude a.

$$E(X(t)) = 0$$

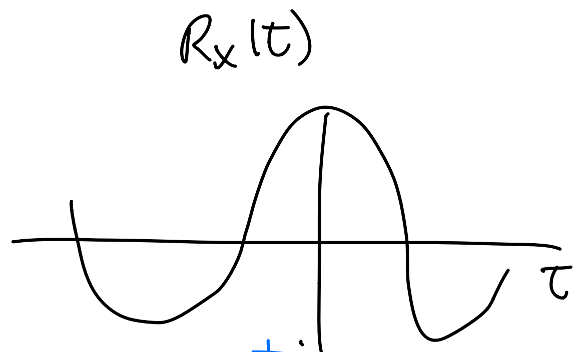
$$R_X\,[X(t)X(t+\tau)] = \frac{a^2}{2}\cos\omega\tau$$

$\Rightarrow$ WSS

$$S_X(f) = \mathcal{F}(R_X(\tau)) = \frac{a^2}{4}\left[\delta\left(f - \frac{\omega}{2\pi}\right) + \delta\left(f + \frac{\omega}{2\pi}\right)\right]$$



signal power focused on a given frequency



symmetric

PSD and LTI Systems (Chapter 10.2)

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

<u>main result</u>

If $X(t)$ is a WSS RP, and it is input to a stable Linear Time Invariant (LTI) system with input response $h(t)$, then the output, $Y(t)$ is also a WSS RP.

<u>And</u> if $X(t)$ has PSD $S_X(f)$ then $Y(t)$ has PSD

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

(equation 10.43)

where $H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi f t}\, dt$

<u>Note</u>: a stable LTI system is one for which

$$\int_{-\infty}^{\infty} |h(t)|^2 \text{ is finite}$$

Example

$X(t)$ is WSS RP

$$m_X = 10 \text{ volts} = E(X(t))$$

LTI impulse response

$$h(t) = \begin{cases} \exp(t/0.2) & 0 \le t \le 0.1 \text{ sec} \\ 0 & \text{else} \end{cases}$$

What is expected value of output $Y(t)$?

$$E(Y(t)) = E\left[\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \underbrace{E(X(t-\tau))}_{}\, d\tau$$

this is a constant, not dependent on t or τ, since X is WSS

$$= E(X(t)) \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$= 10 \int_{0}^{0.1} \exp(t/0.2)\, dt$$

$$= 2\left(e^{1/2} - 1\right) = 1.3 \text{ volts}$$

Proof of WSS of $y(t)$: $y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

we're going to have trouble w/ τ notations, so let's use τ for the autocorrelation, and τ_1 and/or τ_2 for convolution

To show WSS, need

① $m_y(t) = \text{constant}$

$$E(y(t)) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, \underbrace{E(X(t-\tau_1))}_{}\, d\tau_1$$

this is a constant, not dependent on t or τ_1, since X is WSS

$$= m_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1$$

$= m_Y \rightarrow$ a constant not dependent on time

$$\boxed{m_Y = m_X H(0)}$$

just a scaled value of the input's mean

Also need $R_Y(t, t+\tau)$ to be depend only on τ, not t.

$$E(Y(t)\,Y(t+\tau))$$

$$= E\left[\left(\int_{-\infty}^{\infty} h(\tau_1)\,X(t-\tau_1)\,d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2)\,X(t+\tau-\tau_2)\,d\tau_2\right)\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,X(t-\tau_1)\,X(t+\tau-\tau_2)\,d\tau_1\,d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,\underbrace{E\left[X(t-\tau_1)\,X(t+\tau-\tau_2)\right]}_{\text{does not depend on } t,\ \text{because } X(t) \text{ is WSS}}\,d\tau_1\,d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,R_X(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2$$

this depends on τ, but not on t,

so $Y(t)$ is WSS

Power Spectral Density of $Y(t)$

$$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \iiint h(\tau_1)h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2)\, e^{-j2\pi f\tau}\, d\tau_1\, d\tau_2\, d\tau$$

(from previous page)

$$= \iiint h(\tau_1)\, h(\tau_2)\, R_X(\tau')\, e^{-j2\pi f(\tau' + \tau_1 - \tau_2)}\, d\tau_1 d\tau_2 d\tau'$$

(using a change of variables of $\tau' = \tau - \tau_1 + \tau_2$, $d\tau' = d\tau$)

This triple integral is now separable

$$= \left(\int h(\tau_1)\, e^{-j2\pi f\tau_1}\, d\tau_1\right)\left(\int h(\tau_2)\, e^{+j2\pi f\tau_2}\, d\tau_2\right) \cdot \left(\int R_X(\tau')\, e^{-j2\pi f\tau'}\, d\tau'\right)$$

$$= H(f)\, H(-f)\, S_X(f)$$

$$S_Y(f) = |H(f)|^2\, S_X(f)$$