Here we focus on a specific subset of random processes:

stationary random processes (Section 9.6)

But first, a few other specific subsets:

uncorrelated Random Processes

cross-covariance $C_{XY}(t_1, t_2) = 0$ for all t_1 and t_2

orthogonal Random Processes

cross correlation $R_{XY}(t_1, t_2) = 0$ for all t_1 and t_2

Strict-sense stationarity (SSS)

The joint PDFs of <u>any set</u> of samples of the RP does not depend on where you set $t=0$

<u>ex</u>: the joint pdf of $X(t_0)$ and $X(t_1)$ is the same as that of $X(t+t_0)$ and $X(t+t_1)$

Strict-sense stationarity is a very strong constraint, but we've seen examples before:

- flip a coin repeatedly, toss a die
- a sequence of iid random variables..

Some implications of SSS

① $F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x)$

for all t and τ

$\Rightarrow$ The same CDF/PDF characterizes any sample @ any time

② $F_{X(t_1), X(t_2)}(x_1, x_2) = F_{X(0), X(t_2 - t_1)}(x_1, x_2)$

for all t_1 and t_2

$\Rightarrow$ the joint CDF of any 2 samples depends on the time difference between them, not on their individual times

(This generalizes to more than just 2!)

Caution: Stationarity says (loosely) that the <u>randomness</u> of the process doesn't change over time. The process itself might change!

Wide-sense stationarity (wss)

A weaker constraint than sss, for wide-sense stationarity, we only constrain the 1^{st} and 2^{nd} moments

1. The mean function must be constant.

$$m_X(t) = m_X$$

no dependence on time t

2. The autocorrelation is a function of $t_1 - t_2$, but not on either t_1 or t_2 otherwise

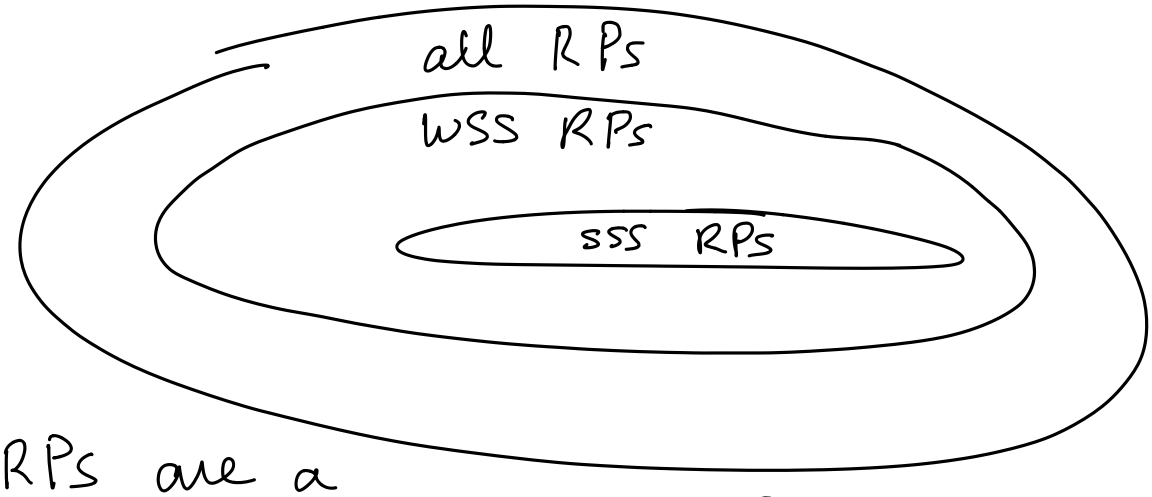
$\Rightarrow$ $R_X(t_1, t_2)$ can be expressed as

$$R_X(t_1 - t_2) = R_X(\tau)$$

Only the time between the samples matters.

WSS RPs can model many real-world problems

ex: A WSS RP $\longrightarrow$ $\boxed{\text{an LTI system}}$ $\longrightarrow$ another WSS RP



SSS RPs are a subset of WSS RPs.

$\Rightarrow$ SSS implies WSS

but WSS does not imply SSS

The 3rd, 4th, 5th moments etc may still depend on t.

$\rightarrow$ Any time you're given $R_X(\tau)$ a function of one time variable this means $X(t)$ is Wide Sense Stationary

And interpret it to be $R_X(\tau) = E(X(t)X(t+\tau))$

Examples: Person's height $\leftarrow$ is this WSS?

is $R_X(3, 30) = R_X(30, 57)$, or in other words,

is $E(X(3)X(30)) = E(X(30)X(57))$?

No $\Rightarrow$ not WSS and $\Rightarrow$ not SSS

Example: Autocorrelation for a discrete-time RP

Let X_n be a sequence of 2 interleaved independent random variables (Example 9.34)

X_n (n odd)
$$p_{X_n}(x) = \begin{cases} 1/2 & x=+1 \\ 1/2 & x=-1 \\ 0 & \text{else} \end{cases}$$

X_n (n even)
$$p_{X_n}(x) = \begin{cases} 9/10 & x=1/3 \\ 1/10 & x=-3 \\ 0 & \text{else} \end{cases}$$

Question 1: is X_n strict sense stationary?

No. That requires the same pmf for every time step n, and that's not true here

Question 2: Is X_n wide sense stationary?

Need to show

1) $m_X(n) = m_X$ for all n

2) $C_X(i,j) = C_X(i-j)$ for all i,j

(or $R_X(i,j) = R_X(i-j)$ for all i,j)

Recall : $m_x(n) = 0$ for all n, and

$$R_x(i,j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1) $m_x(n)$ is constant, same value for all n

so part 1) is true

2) Rewrite $R_x(i,j)$ in terms of $(i-j)$ if possible

$$R_x(i,j) = \begin{cases} 1 & i-j = 0 \\ 0 & i-j \neq 0 \end{cases}$$

there are no terms expressed as "i" or "j", only "$i-j$". So part 2) is also true

$\Rightarrow$ Yes, wide sense stationary

Examples for continuous-time RPs

Random amplitude

$$X(t) = A \cos 2\pi t$$ where A is uniform RV on $[0,1]$

Recall $m_X(t) = \frac{1}{2} \cos 2\pi t$

$$R_X(t_1, t_2) = \frac{1}{3} \cos 2\pi t_1 \cos 2\pi t_2$$

Question 1: is $X(t)$ strict-sense stationary?

No we already showed the pdf of $X(t=0)$ is not the same as, say, the pdf of $X(t=1/2)$

Question 2: is $X(t)$ wide sense stationary?

No. $m_X(t)$ is not constant with time.

Random phase RP

$$X(t) = \cos(\omega t + \theta)$$

where θ is a uniform RV on $[-\pi, \pi]$

Recall

$$m_X(t) = E(X(t)) = 0$$

and $R_X(t_1, t_2) = E(X(t_1)X(t_2))$

$$= \frac{1}{2}\cos(\omega(t_1 - t_2))$$

Is this RP WSS?

a) mean is a constant that does not depend on time. Good so far

b) autocorrelation function depends on $(t_1 - t_2)$ not on t_1 or t_2 individually.

$$R_X(t_1, t_1 + \tau) = \frac{1}{2}\cos(\omega(t_1 - (t_1 + \tau)))$$

$$= \frac{1}{2}\cos \omega\tau$$

$$= R_X(\tau)$$

Yes, both conditions are satisfied $\Rightarrow$ WSS.

An example that requires interpreting a given $R_X(\tau)$.

If $R_X(\tau) = e^{-|\tau|}$ for a WSS RP $X(t)$,

what is $E(X(2)X(4))$?

$$E(X(2)X(4)) = R_X(2,4)$$

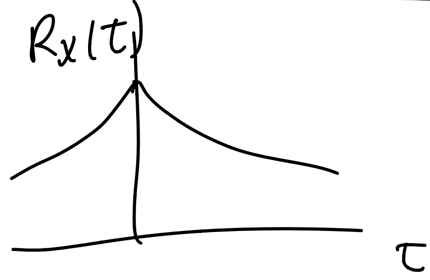
$$= R_X(2-4) = R_X(4-2)$$

$$= e^{-2}$$

A few sample autocorrelation functions $R_X(\tau)$

$$R_X(\tau) = e^{-a|\tau|}$$

for all τ and $a > 0$



Random phase sinusoid

$$R_X(\tau) = \frac{a^2}{2} \cos 2\pi f_0 \tau \qquad \text{for all } \tau$$

when $X(t)$ is zero mean.

This $R_X(\tau)$ is periodic w/period $1/f_0$

If X_n is iid sequence of RVs with zero mean and variance σ^2

and $Y_n = \dfrac{X_n + X_{n-1}}{2}$ (the average)

First, $R_X(i,j) = R_X(i-j) = \begin{cases} \sigma^2 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} = E(X_i X_j)$

Then $E(Y_n) = 0$ and $C_Y(i,j) = R_Y(i,j)$

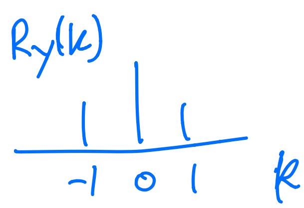
$$R_Y(i,j) = E(Y_i Y_j) = \frac{1}{4} E\left[(X_i + X_{i-1})(X_j + X_{j-1})\right]$$

$$= \frac{1}{4} E[X_i X_j] + \frac{1}{4} E[X_i X_{j-1}] + \frac{1}{4} E[X_{i-1} X_j] + \frac{1}{4} E[X_{i-1} X_{j-1}]$$

Now substitute for $R_X(i-j)$

$$R_Y(i-j) = \frac{1}{4}\left[R_X(i-j) + R_X(i-j+1) + R_X(i-1+j) + R_X(i-j)\right]$$

$$= \frac{1}{4} \begin{cases} 2\sigma^2 & \text{if } i=j \\ \sigma^2 & \text{if } |i-j| = 1 \\ 0 & \text{else} \end{cases}$$



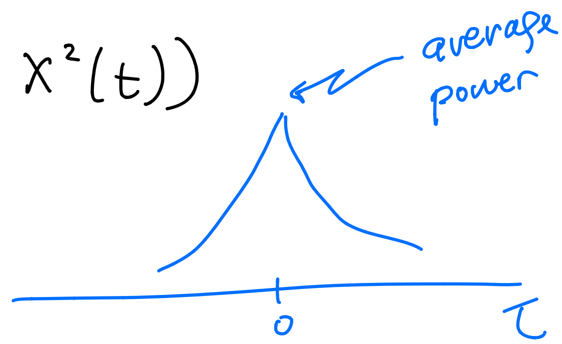
4 of 5 Properties of WSS and $R_X(\tau)$

(Note: if $C_X(t_1, t_2) = C_X(t_2 - t_1) = C_X(\tau)\ \forall t_1, t_2$ and $m_X(t) = m_X$, then $R_X(t_1, t_2) = R_X(\tau)$)

① $R_X(0)$ = average power of $X(t)$

$$= E(X(t)X(t)) = E(X^2(t))$$

$$\boxed{R_X(0) > 0}$$



② $R_X(-\tau) = R_X(\tau)$ symmetric

proof:

$$R_X(\tau) = E(X(t)X(t+\tau))$$

$$= E(X(t+\tau)X(t))$$

let $\tau' = t + \tau$

$$= E(X(\tau')X(\tau' - \tau))$$

$$= R_X(-\tau)$$

③ $|R_X(\tau)| \leq R_X(0)$ maximum @ $\tau = 0$

④ Positive semidefinite function

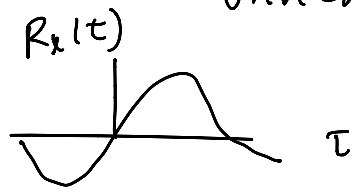
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1) R_X(t_1 - t_0) a(t_0)\, dt_1\, dt_0 \geq 0$$

for any real function $a(t)$

Examples of applying the 1st 4 properties

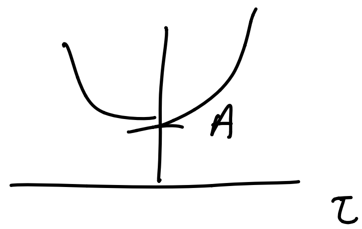
Are the following functions valid autocorrelation functions?

a) $R_X(\tau) = A \sin w\tau$



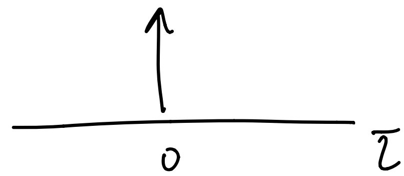
No. not even symmetric

b) $R_X(\tau) = A e^{|\tau|}$



No. $R_X(0)$ is not max.

c) $R_X(\tau) = A\delta(t)$



Yes. Satisfies all.

This is the autocorrelation function for white noise

(Any 2 samples are orthogonal if $t_1 \neq t_2$)

d) $R_X(\tau) = A \cos w\tau$

Yes. Even symmetric, max at zero, positive nonzero $R_X(0)$.

Also positive semi definite:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1)\, A \cos w(t_1 - t_0)\, a(t_0)\, dt_1\, dt_0$$

$$= \iint a(t_1)\, A \cos wt_1 \cos wt_0\, a(t_0)\, dt_1\, dt_0$$

$$= A\left[\quad\right]\left[\quad\right] = A\left[\quad\right]^2$$

which is positive.

So Yes

Property 5 of $R_X(\tau)$

$$P(|X(t+\tau) - X(t)| > \varepsilon) \le \frac{2(R_X(0) - R_X(\tau))}{\varepsilon^2}$$

The autocorrelation function measures the rate of change of the WSS RP

Proof: uses the Markov inequality from section 4.6 equation (4.75)

$P(X \ge a) \le \frac{E(X)}{a}$ for non-negative RVs X.

Think of $|X(t+\tau) - X(t)|^2$ as the RV and apply Markov inequality for ε

$$P(|X(t+\tau) - X(t)| > \varepsilon)$$

$$= P\left((X(t+\tau) - X(t))^2 > \varepsilon^2\right)$$

$$\le \frac{E\left((X(t+\tau) - X(t))^2\right)}{\varepsilon^2}$$

$$= \frac{E(X^2(t+\tau)) + E(X^2(t)) - 2E(X(t+\tau)X(t))}{\varepsilon^2}$$

$$= \frac{2[R_X(0) - R_X(\tau)]}{\varepsilon^2}$$

Interpretation of Property 5 of $R_X(\tau)$

$$P\left(|X(t+\tau)-X(t)|>\varepsilon\right) \leq \frac{2(R_X(0)-R_X(\tau))}{\varepsilon^2}$$

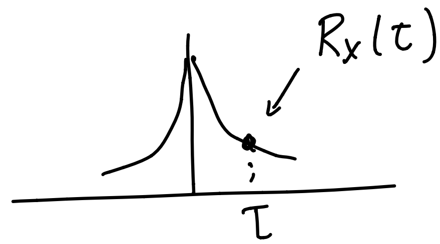
Interpretation: Relates 2 things

1. the amount X varies in τ seconds

$$|X(t+\tau)-X(t)|$$

2. the autocorrelation function @ τ and 0.

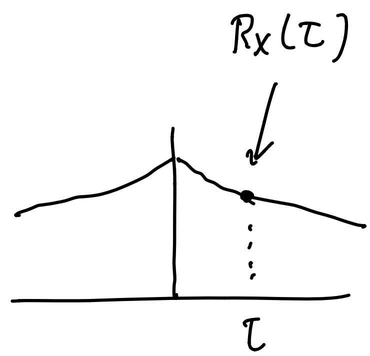
a) if $R_X(0)-R_X(\tau)$ is large



then R_X decays rapidly
and the probability of a large change in X is high.

$\Rightarrow$ what happens at time t tells us little about what will happen at $t+\tau$

b) if $R_X(0)-R_X(\tau)$ is small



then R_X decays slowly
and the probability of a large change in X is small

$\Rightarrow$ what happens at time t tells us a lot about what will happen at $t+\tau$

Here, $|X(t+\tau)-X(t)|>\varepsilon$ represents a large change in X between time t and time $t+\tau$

Application of $R_X(\tau)$ for prediction

(similar concepts for discrete time)

If a WSS RP $X(t)$ has an autocorrelation function $R_X(\tau)$, and we want to predict the future value(s) of $X(t)$ from the current/previous values, what's the best predictor?

Example: Suppose we want to predict $X(t+1)$ from $X(t)$, and we want to minimize the mean squared error and we want a linear predictor

Define predictor as $\hat{X}(t+1) = a\,X(t)$

what's the best value of a?

Mean squared error: $E\left[\left(\hat{X}(t+1) - X(t+1)\right)^2\right]$

$$= E\left[\left(a\,X(t) - X(t+1)\right)^2\right]$$

$$= E\left[a^2 X(t)^2 - 2a\,X(t)X(t+1) + X(t+1)^2\right]$$

$$= a^2 E(X(t)^2) - 2a E(X(t)X(t+1)) + E(X(t+1)^2)$$

translate this into $R_X(\tau)$

$$= a^2 R_X(0) - 2a R_X(1) + R_X(0)$$

$$= (a^2+1) R_X(0) - 2a R_X(1)$$

This is a function of prediction coefficient a

Find the best a by differentiating w.r.t a setting to zero, and solving for a.

$$\frac{d}{da} = 2a R_X(0) - 2 R_X(1) = 0$$

$$\boxed{a = \frac{R_X(1)}{R_X(0)}}$$