Models for random processes (Section 9.2)

(to describe how $X(t_1)$ is related to $Y(t_2)$)

Recall:

1. Fix outcome → a function of time

   Use tools from ECE 301.

2. Fix time → a random variable

   Use tools already learned in ECE 302

3. Fix neither.

What can we say about the random signal?

Do we have any useful tools already? (Yes)

How should we extend these to be relevant to this new more complicated situation?

Some thoughts:

PDF of $X(t)$

a pdf at each instant in time

– is it constant or does it vary with time?

Joint PDF of $X(t_1)$ and $X(t_2)$

how is the pdf at one instant t_1 related to the pdf at another instant t_2?

Can plausibly be extended to n samples in time

– BUT messy – there are an infinite # samples!

A different less complete modeling approach:

Moments of a Random Process

- an extension of the moments of a RV
- moments are now a function of Time

1st moment: mean $m_X(t) = E(X(t))$

2nd moment: $E(X^2(t))$

Variance: $Var(X(t)) = E(X^2(t)) - E(X(t))^2$

But there are other useful moments.

Autocorrelation, cross correlation, auto covariance, cross covariance

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Suppose we sample $X(t)$ at 2 time instants t_1 and $t_2 = t_1 + \tau$.

We have 2 RVs: $X(t_1)$ and $X(t_2) = X(t_1 + \tau)$

The correlation between these two, $E(X(t_1)X(t_2))$, is important enough we give it a name autocorrelation and a notation $R_X(t_1, t_2)$

$$= R_X(t_1, t_1 + \tau)$$

This gives us information about the relationship between 2 time samples of the same random process

We can also consider the covariance between $X(t_1)$ and $X(t_2)$

$$C_X(t_1, t_2) = \mathrm{Cov}(X(t_1), X(t_2))$$
$$= E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))]$$
$$= E(X(t_1)X(t_2)) - E(X(t_1))E(X(t_2))$$

This is called the auto covariance function

Relationships:

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)\, m_X(t_2)$$

$$C_X(t_1, t_1) = \mathrm{Var}(X(t_1))$$

Note: the autocorrelation, autocovariance, and mean functions do not completely characterize a RP.

But, they do describe some basic information about $X(t)$

(we can denote these as $C_X(t, t+\tau)$ and $R_X(t, t+\tau)$)

Next, consider two random processes

$X(t)$ and $Y(t)$

and we sample one at time t_1

and another at time $t_1 + \tau$

We can compute the correlation and covariance between these two RVs,

$X(t_1)$ and $Y(t_1+\tau)$

Cross correlation

$$R_{xy}(t_1, t_1+\tau) = E(X(t_1)\,Y(t_1+\tau))$$

cross covariance

$$C_{xy}(t_1, t_1+\tau) = E(X(t_1)\,Y(t_1+\tau)) - E(X(t_1))\,E(Y(t_1+\tau))$$

$$= R_{xy}(t_1, t_1+\tau) - m_x(t_1)\,m_y(t_1+\tau)$$

Examples about 2nd order statistics

Suppose two RPs, $X(t)$ and $Y(t)$

Example:

$X(t)$ is height of a person over their life time. (Each person will have a different <u>instance</u> of the RP.)

what is the correlation between their height at 3 years and at 30 years?

$$R_X(3, 30) = E[X(3)X(30)]$$

one point on the autocorrelation function

Example:

$X(t)$ is the size of a dog and

$Y(t)$ is the size of a dogs paws

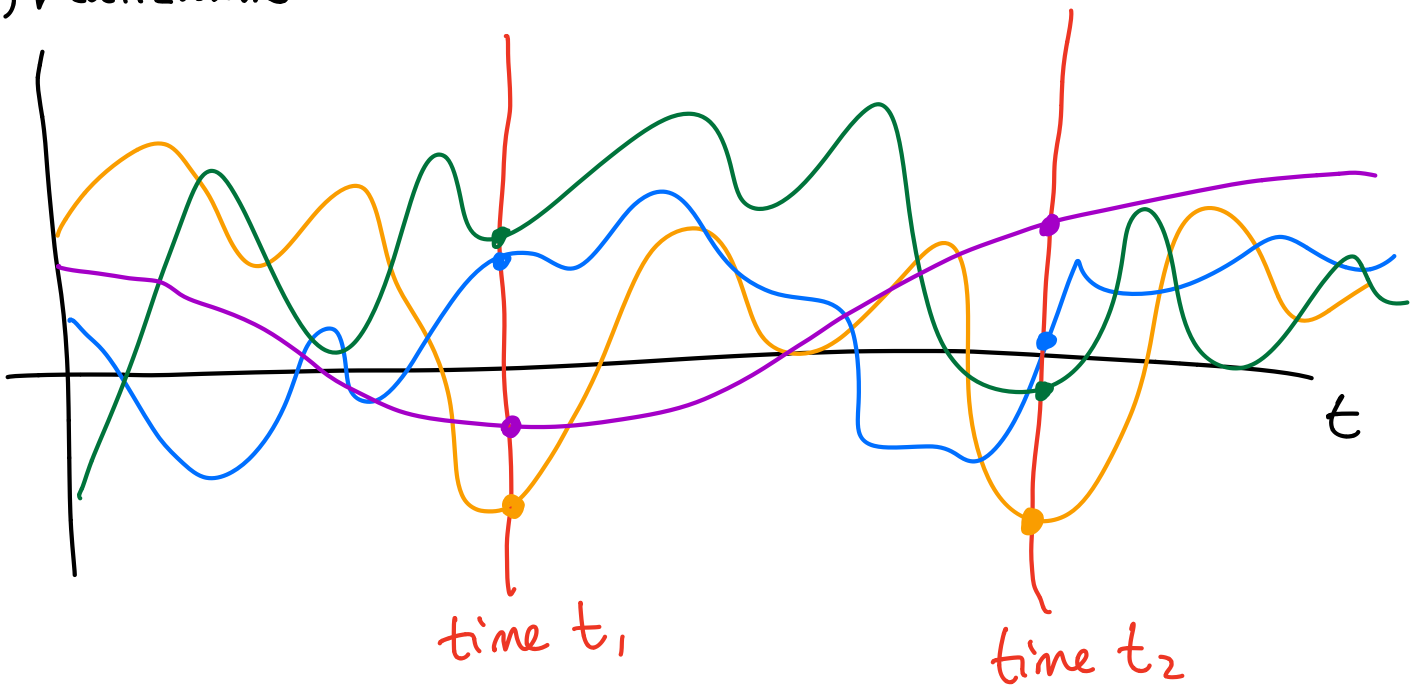
what is the correlation between the size of the paws at 3 months and the size of the dog at 2 years

$$R_{XY}(2, 0.25) = E(X(2)\,Y(0.25))$$

one point on the cross correlation function

Concept of auto correlation visually

X(t) realizations

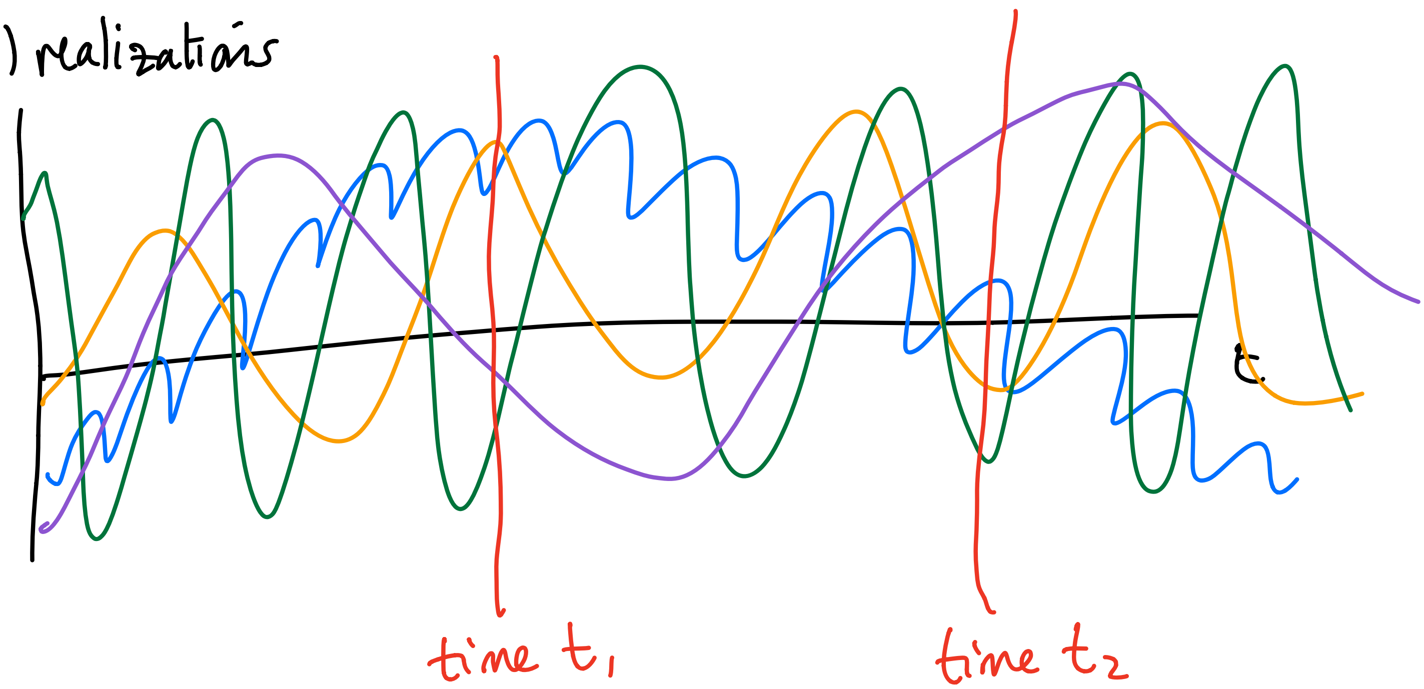


$E(X(t_1))$ = mean of all possible realizations at time t_1

$R_X(t_1, t_2) = E(X(t_1)\,X(t_2))$

= expectation over all realizations

Y(t) realizations

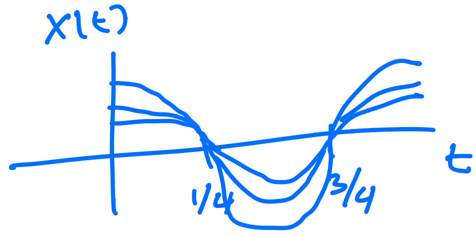


$R_{XY}(t_1, t_2) = E(X(t_1)\,Y(t_2))$ relates X at t_1 to Y at t_2

Computing auto correlation : example

Suppose A is a uniform RV on $[0,1]$

and $X(t) = A \cos 2\pi t$



$$R_X(t_1, t_2) = E\left(X(t_1) X(t_2)\right)$$

$$= E\left[\left(A \cos 2\pi t_1\right)\left(A \cos 2\pi t_2\right)\right]$$

$$= E(A^2) \cos 2\pi t_1 \cos 2\pi t_2$$

Because A is random, and $\cos 2\pi t$ is NOT

$$\boxed{E(A^2) = \int_0^1 x^2 f_A(x)\,dx = \frac{x^3}{3}\Big|_0^3 = \frac{1}{3}}$$

$$\Rightarrow R_X(t_1, t_2) = \frac{1}{3} \cos 2\pi t_1 \cos 2\pi t_2$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) - E(X(t_1)) E(X(t_2))$$

$$= \frac{1}{3} \cos 2\pi t_1 \cos 2\pi t_2$$

$$- \frac{1}{4} \cos 2\pi t_1 \cos 2\pi t_2$$

$$= \frac{1}{12} \cos 2\pi t_1 \cos 2\pi t_2$$

recall $E(X(t_1)) = E(A) \cos 2\pi t_1 = \frac{1}{2} \cos 2\pi t_1$

Computing cross-correlation : example

$X(t) = \cos(wt + \theta)$

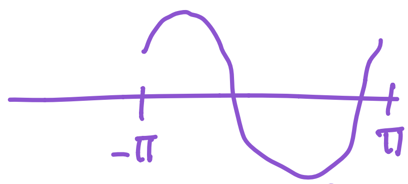
$Y(t) = \sin(wt + \theta)$

Both depend on θ, a uniform RV on interval $(-\pi, \pi]$

$f(\theta) = \frac{1}{2\pi}$ when $\pi < \theta \leq -\pi$

$$E(X(t)) = E(\cos(wt + \theta))$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(wt + \theta)\, d\theta$$



(one full cycle)

$$= 0$$

and $E(Y(t)) = 0$ also

$$\Rightarrow C_X(t_1, t_2) = R_X(t_1, t_2)$$

auto correlation

$$R_X(t_1, t_2) = E(X(t_1)X(t_2))$$

$$= E(\cos(wt_1 + \theta)\cos(wt_2 + \theta))$$

$$= E\left(\tfrac{1}{2}\cos(w(t_1 + t_2) + 2\theta) + \tfrac{1}{2}\cos(w(t_1 - t_2))\right)$$

(see identities on next page)

$$= \frac{1}{2}\cos(w(t_1 - t_2))$$

(1^{st} term is similar to computing mean, but there are two complete cycles)

Trig Identities

Recall: $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

Cross correlation

$$R_{xy}(t_1,t_2) = E\left(X(t_1)\,Y(t_2)\right)$$

$$= E\left[\cos(\omega t_1+\Theta)\sin(\omega t_2+\Theta)\right]$$

$$= E\left(\frac{1}{2}\sin\left(\omega(t_1+t_2)+2\Theta\right) - \frac{1}{2}\sin\left(\omega(t_1-t_2)\right)\right)$$

First term, taking expectation over Θ, goes to zero. Second term is constant.

$$= -\frac{1}{2}\sin\left(\omega(t_1-t_2)\right)$$

and for these 2 RPs, $C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2)$

Note that this depends on t_1-t_2, but does not otherwise depend on either t_1 or t_2 individually

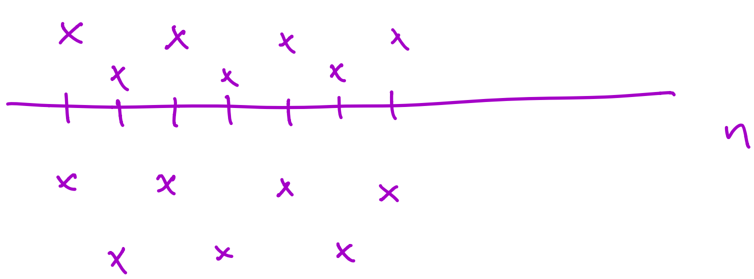
Example: Autocorrelation for a discrete-time RP (Example 9.34)

Let X_n be a sequence of 2 interleaved independent random variables

X_n (n odd) $\quad p_{X_n}(x) = \begin{cases} 1/2 & x = +1 \\ 1/2 & x = -1 \\ 0 & \text{else} \end{cases}$

X_n (n even) $\quad p_{X_n}(x) = \begin{cases} 9/10 & x = 1/3 \\ 1/10 & x = -3 \\ 0 & \text{else} \end{cases}$

Possible values for this Random Process



Find mean and autocorrelation and autocovariance

Mean: $m_X(n) \equiv m_n = E(X_n)$

for n odd, $m_X(n) = \frac{1}{2}(+1) + \frac{1}{2}(-1) = 0$

for n even, $m_X(n) = \frac{9}{10}\left(\frac{1}{3}\right) + \frac{1}{10}(-3) = 0$

so $m_X(n) = 0$ for all n

Autocovariance $C_X(i,j) = E(X_i X_j) - E(X_i)E(X_j)$

$= E(X_i X_j) = R_X(i,j)$

since $E(X_i) = 0$ for all i

Autocorrelation $R_X(i,j) = E(X_i X_j)$

Case 1: $i=j$ $\quad R_X(i,i) = E(X_i^2)$

for $i=j$ even, $\quad E(X_i^2) = \frac{9}{10}\left(\frac{1}{3}\right)^2 + \frac{1}{10}(-3)^2$

$= \frac{9}{90} + \frac{9}{10} = \frac{1}{10} + \frac{9}{10} = 1$

for $i=j$ odd, $\quad E(X_i^2) = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$

So $R_X(i,i) = 1$ for all i

Case 2: $i \neq j$ $\quad E(X_i X_j) = E(X_i)E(X_j)$

since X_i and X_j are independent if $i \neq j$

So $R_X(i,j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = C_X(i,j)$

Review — Random (stochastic) processes
using discrete time notation

$X(n, \omega)$ or $X_n(\omega)$ depends on both the outcome ω and the time n.

$x(n)$ or X_n is an abbreviation that hides dependency on ω

X_n is also a random variable, a sample of the random process at n.

If we fix ω, then X_n is a deterministic function of time ⇒ tools from ECE 301

If we fix n, then X_n is a random variable ⇒ tools from this class

$E(X_n) = m(n)$ or $= m_n$ → mean is a function of time

If we don't fix either ω or n, we need new tools.

Here we focus on 2nd-order statistics and how they vary as a function of time

Autocorrelation $R_X(i,j) = E(X_i X_j)$

Autocovariance $C_X(i,j) = R_X(i,j) - m_X(i)\, m_X(j)$

Cross correlation $R_{XY}(i,j) = E(X_i Y_j)$

Cross covariance $C_{XY}(t_1,t_2) = R_X(i,j) - m_X(i)\, m_Y(j)$