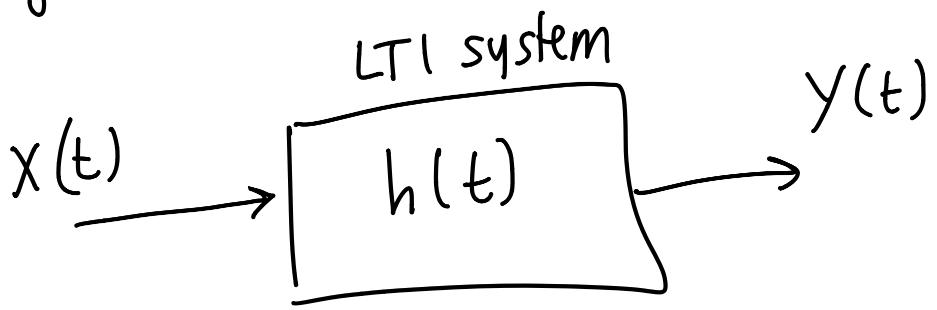
Introduction to Random Processes

Topic 4.1. Examples of $X(t)$ and X_n
mean and variance functions

Topic 4.2 How are $X(t_1)$ and $Y(t_2)$ related?
2nd-order statistics
ex: auto correlation
and cross-covariance

Topic 4.3 Defining a useful subset of RP
using their auto correlation function

The goal



what is the relationship between
$X(t)$ and $Y(t)$?

Chapter 9: Random Processes

The outcome of a random experiment that varies as a function of time (and/or space)

<u>Examples</u>: speech signals
images
temperature and the demand for power

Overview of topics:

Sect 9.1: definitions
Sect 9.2: joint distributions, mean function, autocorrelation function, autocovariance function
Sect 9.6: Stationary Random Processes

Sect 10.1: Power Spectral Density
Sect 10.2: Response of an LTI system when the input is a Random Process

Stochastic Process is another name for a Random Process

Abbreviation: RP

Recall: We do 4 things in this class

1. build models
2. compute probabilities
3. learn or infer
4. Compute summary statistics.

With respect to random processes,
we will focus on building models and
computing summary statistics.

Specifically, we will build on the concepts
of correlation and covariance
to consider autocorrelation and
autocovariance, and
cross correlation and
cross covariance.

We will also build models by considering
what happens when we input a
specific type of random process into
a Linear Time Invariant system (LTI)

Questions we may want to address about Random Processes

mean – how does it change with time?

Variance – " " "

How does a sample at one time instant relate to a sample @ another time?

Does each sample have the same PDF, or does the PDF vary with time?

Can we compute a time average and expect it to represent the average @ a specific time?

$\boxed{\text{How can we make decisions under uncertainty?}}$

So far, we've chosen an instant in time.

But the world is dynamic.

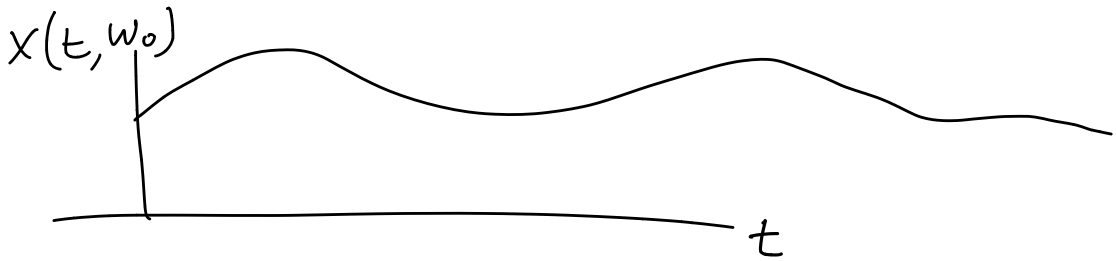
For engineering, we need to quantify these things, need to have a mechanism to analyze

Random (Stochastic) Processes (RP)

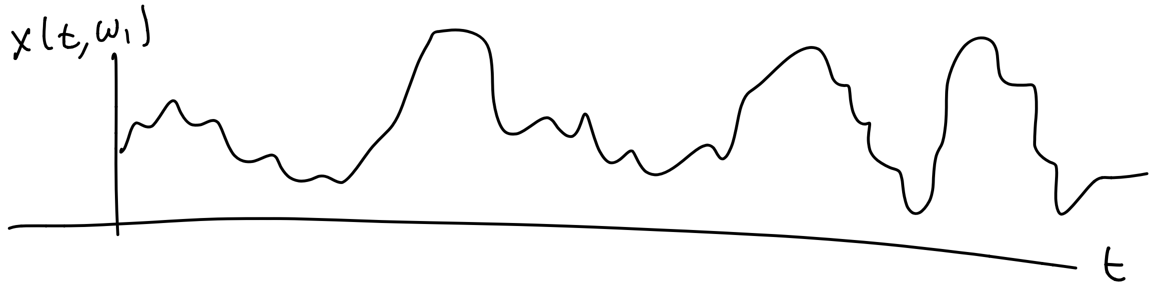
A random experiment, a set of possible outcomes ω, and - for each possible outcome - an instance of the RP: $X(t,\omega)$

$X(t,\omega)$ is a random process - a deterministic function of both t and ω

For one outcome



For another outcome

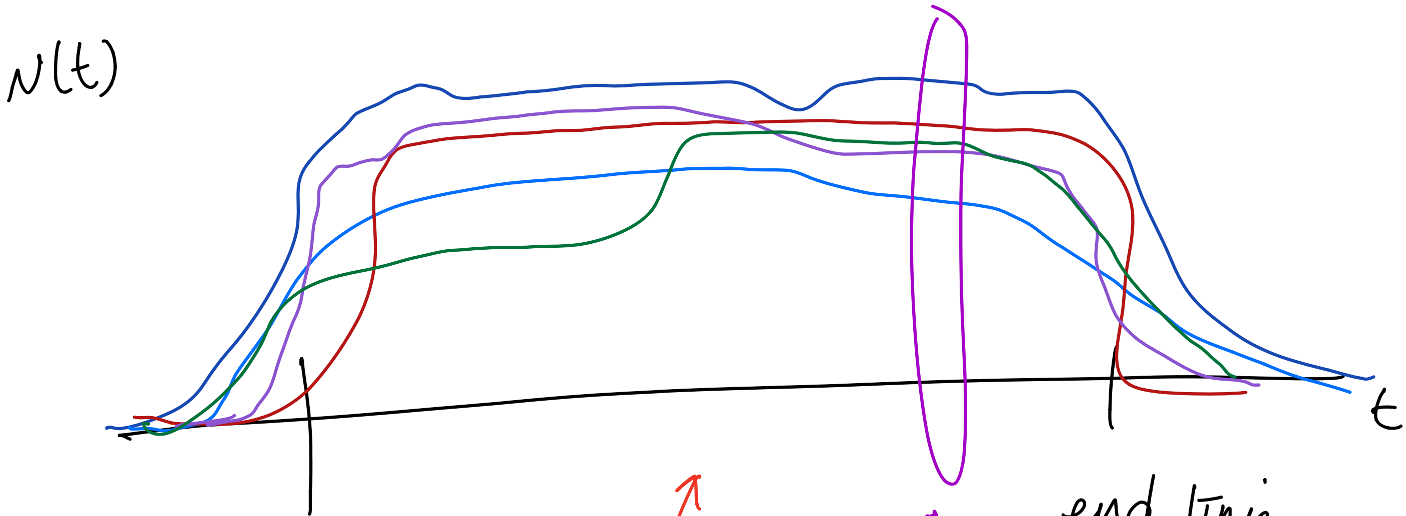


As a 2D function, we can explore it in t, ω, and both

1. Fix $t=t_0$. Sample in time. (use 302 methods)
   $X(t_0,\omega)$ is a random variable
2. Fix $\omega=\omega_0$. Choose a realization/instance/outcome
   $X(t,\omega_0)$ is just a function of time (use 301 methods)
3. New tools when we fix neither ω nor t.

An example random process:

the number of ECE 302 students in the classroom as a function of time. $N(t)$



start time of class

end time of class

at this snapshot in time, we have a random variable

overall, this is a set of realizations

Random process examples

Context: driving on an icy day on the streets of West Lafayette IN

accidents as a function of time

stopping time as a function of space

traction of one vehicle as a function of time

distance of slippage as f(t)

thickness of ice on the windsheild

precipitation rate as function time

cars that pass through Grant/State intersection

cars that are waiting at a specific red light

wait time of a particular car @ a red light

pedestrians in a walkway @ an intersection

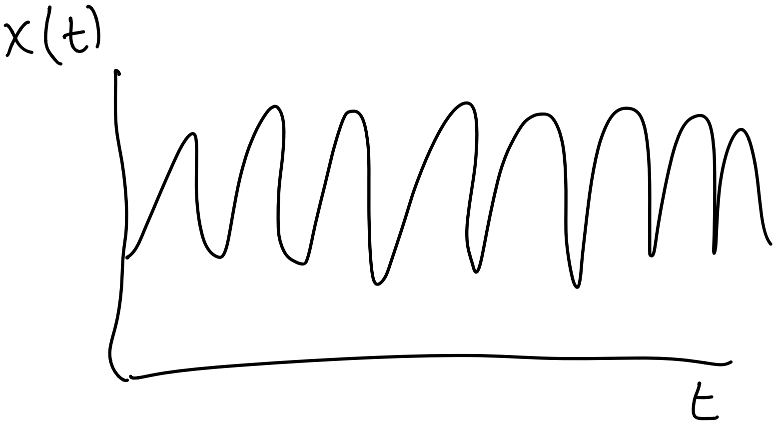
Color of the light being emitted from the traffic signal

RPs can be discrete or continuous time, and can have discrete or continuous values

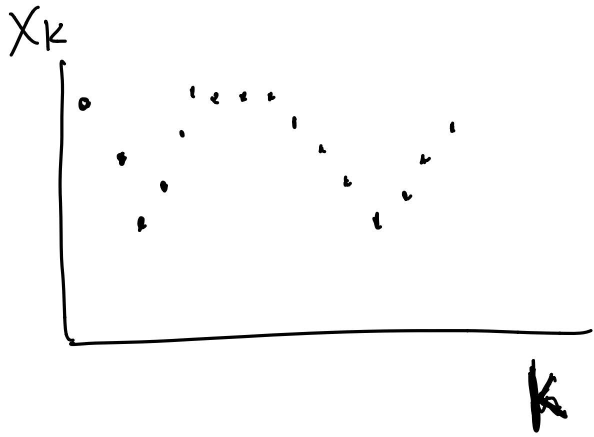
Examples :

(one sample instance each)

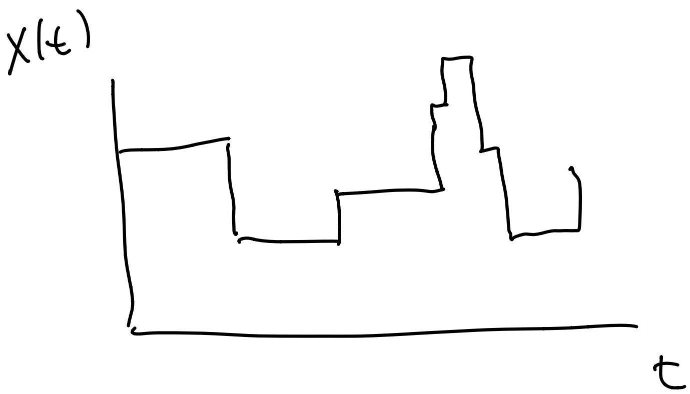
Continuous time
Continuous value



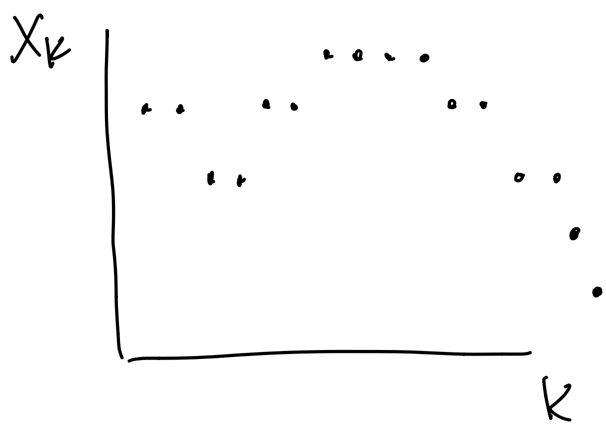
Discrete time
Continuous value



Continuous time
Discrete value



Discrete time
Discrete value



Continuous-time continuous-value examples:

1. random amplitude sinusoid
2. random phase sinusoid
3. random frequency sinusoid

Note: $X(t)$ is both the name of the RP (a function of t)
and the name of the RV that corresponds to sampling $X(t)$ at t.

Random Amplitude.

Let A be uniformly distributed on $[0,1]$.
(Recall its full name is $A(\omega)$, a R.V.)

Let $X(t) = A \cos 2\pi t$ (we could call it $X(t,\omega)$)

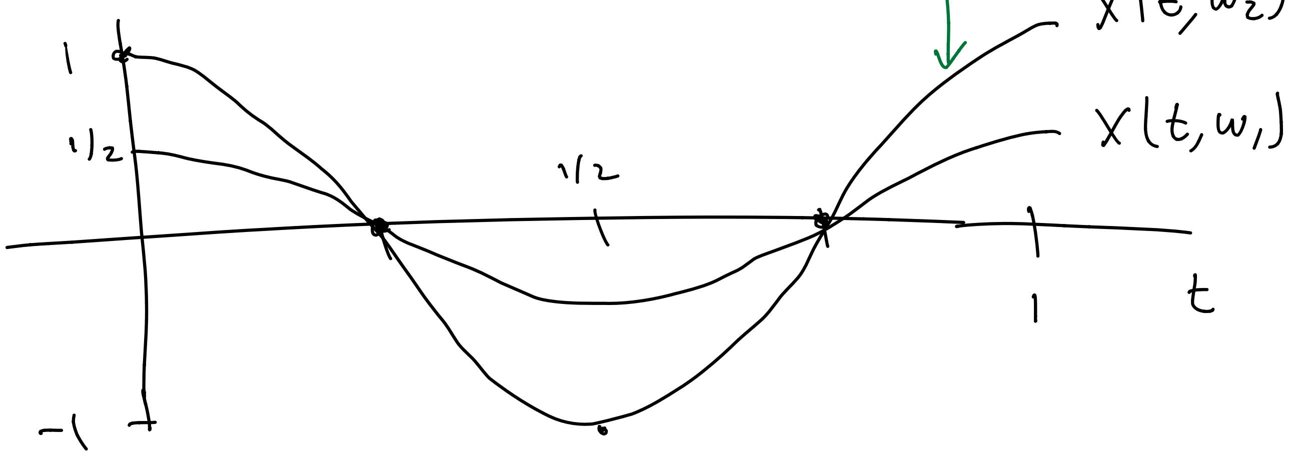
if $X(\omega_1) = 1/2$

$$x(t, \omega_1) = \frac{1}{2} \cos 2\pi t$$

if $X(\omega_2) = 1$

$$X(t, \omega_2) = \cos 2\pi t$$

different possible realizations



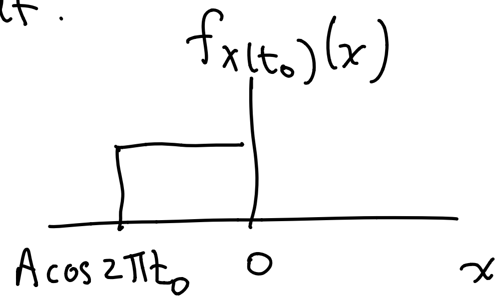
We call the set of all possible outcomes the ensemble.

Recall: we can view this RP 3 ways

① Fix $t = t_0$. $X(t_0, w)$ is a random variable.

w/ possible values between 0 and $A \cos 2\pi t_0$

example pdf:



(if $\cos 2\pi t_0$ is negative)

A different RV for each value of t

Note: for $t_0 = 1/4$ or $t_0 = 3/4$,

there is only one possible outcome for

$X(t_0) = A \cos 2\pi/4 = 0.$

② Fix $w = w_0$. $X(t, w_0)$ is a function of time.

(Two examples were shown earlier)

③ Fix neither. Compute mean and variance as a function of time

$$E(X(t)) = E(\underbrace{A}_{\text{random part}} \underbrace{\cos 2\pi t}_{\text{deterministic part}}) = E(A) \cos 2\pi t$$

expectation over outcomes w. Not a time average

$$= \underbrace{\frac{1}{2} \cos 2\pi t}_{\text{the mean is a function of time.}}$$

$$= m(t) = \mu(t)$$

$$Var(X(t)) = E(X(t)^2) - E(X(t))^2$$

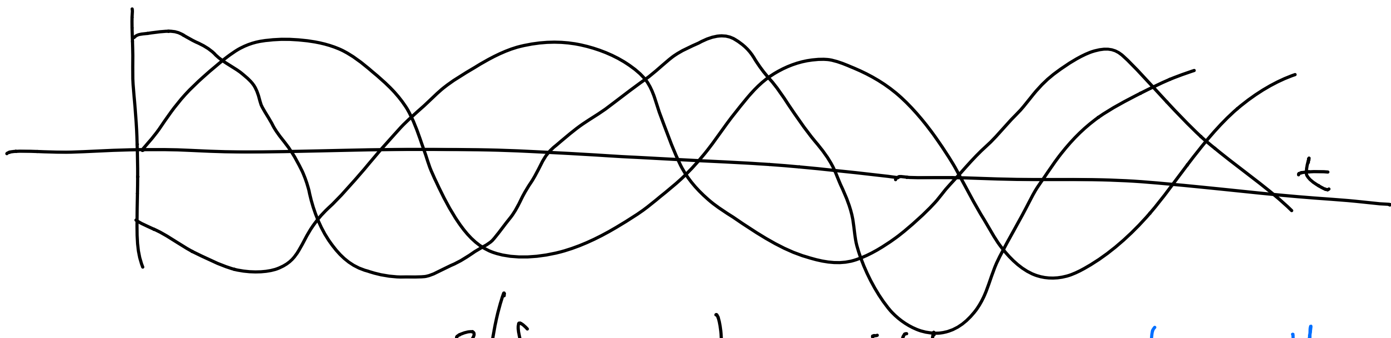
$$E(X(t)^2) = E(A^2 \cos^2 2\pi t) = \frac{1}{3} \cos^2 2\pi t$$

$$\boxed{Var(X(t)) = \frac{1}{12} \cos^2 2\pi t}$$

Another example: Random Phase Sinusoid

Let $\Theta(\omega)$ be a uniform RV on $[0, 2\pi)$.
and let B be a constant.

Let $Y(t,\omega) = B\cos(2\pi t + \Theta)$



3 (of many) possible sample paths for this RP

$$E(Y(t)) = E\left(B\cos(2\pi t + \Theta)\right)$$

$$= B\,E(\cos(2\pi t + \Theta))$$

$$= B\cdot 0 = 0$$

$$E(Y^2(t)) = E\left(B^2(\cos(2\pi t + \Theta))^2\right)$$

$$= \frac{1}{2}B^2 E(1 + \cos(4\pi t + 2\Theta))$$

$$= B^2/2 + 0$$

$\cos^2\Theta = \frac{1}{2}[1 + \cos 2\Theta]$

$$\mathrm{Var}(Y(t)) = \frac{B^2}{2} - 0 = B^2/2$$

Another example: Random frequency sinusoid

Roll a die. Sample space $S = \{1, 2, 3, 4, 5, 6\}$

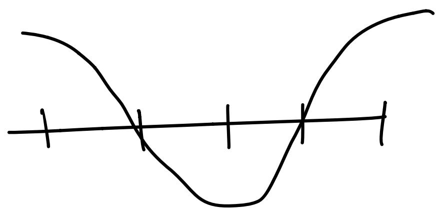
k = # on die.

The Random Process is $X(t, k) = \cos 2\pi k t$

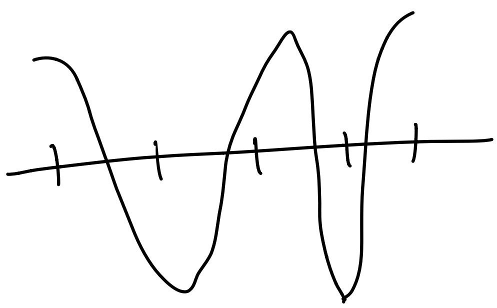
$k = 1, 2, 3, .., 6$

This random process has its frequency selected @ random.

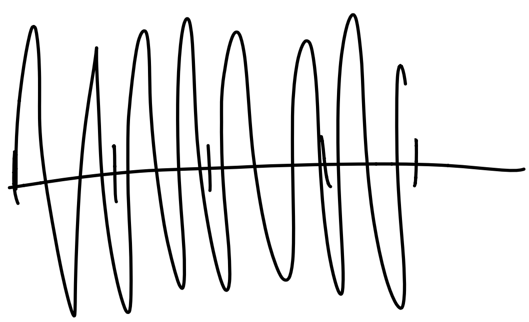
$\omega = k$	$X(t)$
1	$\cos 2\pi t$
2	$\cos 4\pi t$
3	$\cos 6\pi t$
4	$\cos 8\pi t$
5	$\cos 10\pi t$
6	$\cos 12\pi t$



$k=1$



$k=2$



$k=6$

These 3 RPs (random amplitude, random phase, and random frequency) are common in communication theory. (ECE 440)

Satellite TV, cell phones, AM and FM radios

Another example RP: QPSK

Quaternary Phase Shift Keying (for comm systems)

4 equally probable symbols (s_0, s_1, s_2, s_3)

One sent every T seconds

To send symbol s_k, send the waveform

$$x(t, s_k) = \cos(2\pi f_0 t + \pi/4 + k\pi/2)$$

during each T-second interval

Because there is randomness in which symbol is sent, $X(t)$ is a random process

Receive: $Y(t) = X(t) + N(t)$ ← also a RP

Goal is to estimate the transmitted $X(t)$ from the received $Y(t)$, or more precisely, estimate s_k from $Y(t)$ in each time interval

QPSK is used for satellite transmission of MPEG-2 video

A time delayed ramp function

$$X(t) = t - T \quad , \text{ where } T \text{ is a RV,}$$

$$f_T(t) = e^{-t} \qquad t \geq 0$$

what is PDF of the RV $X(t)$?

(For clarity and the sake of explanation, let's look for the PDF of a specific $X(t_0)$ and then generalize)

$$F_X(x) = P(X(t_0) \leq x) = P(t_0 - T \leq x)$$

$$= P(T \geq t_0 - x)$$

At this stage, we have to be careful. $T \geq 0$, so

$$F_X(x) = 1 - F_T(t_0 - x) \quad \text{if} \quad t_0 - x > 0 \quad \text{or} \quad x < t_0$$

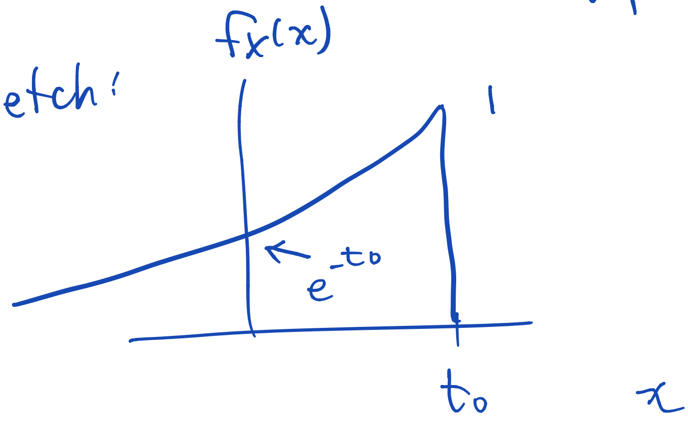
and $F_X(x) = 1 - 0 = 1$ if $t_0 - x < 0$ or $x \geq t_0$

Note: this CDF has no discontinuities! so no special cases to consider when differentiating

$$f_X(x) = \frac{d}{d} F_X(x) = -f_T(t_0 - x)(-1) \quad \text{when} \quad x < t_0$$

$$= f_T(t_0 - x) = \begin{cases} e^{x - t_0} & x < t_0 \\ 0 & \text{else} \end{cases}$$

sketch:



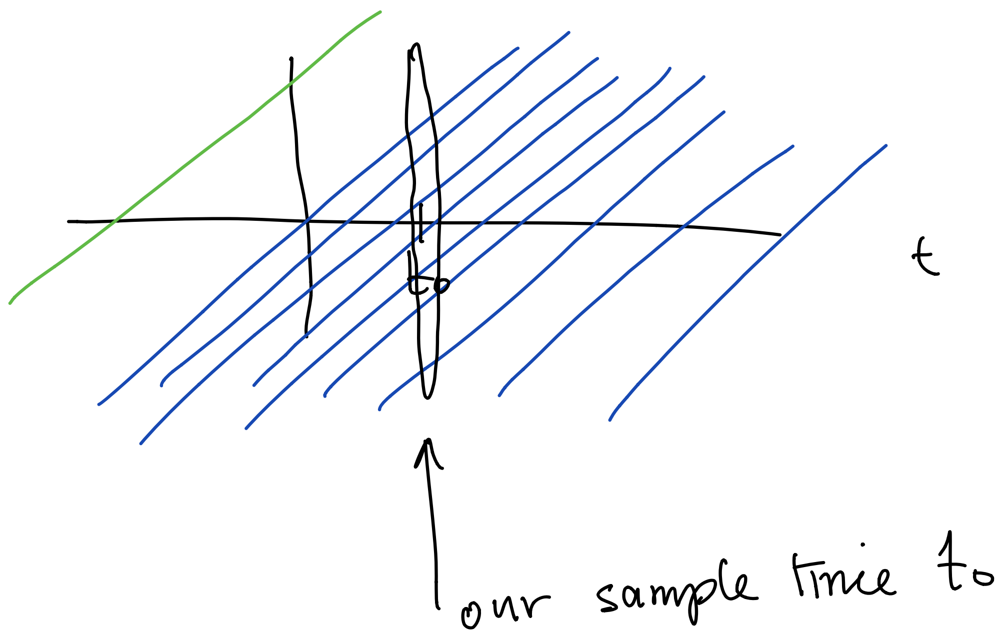
Intuition:

what are the possible realizations of the RP?

Because $T \geq 0$, the ramp can become positive for any positive value of time.

The blue ramps below are all possible.

The green ramp is not.



Not possible to get values of $X(t_0) > t_0$

"most likely" scenario is to have $X(t_0) = t_0$ because the "most likely" value of $T = 0$.

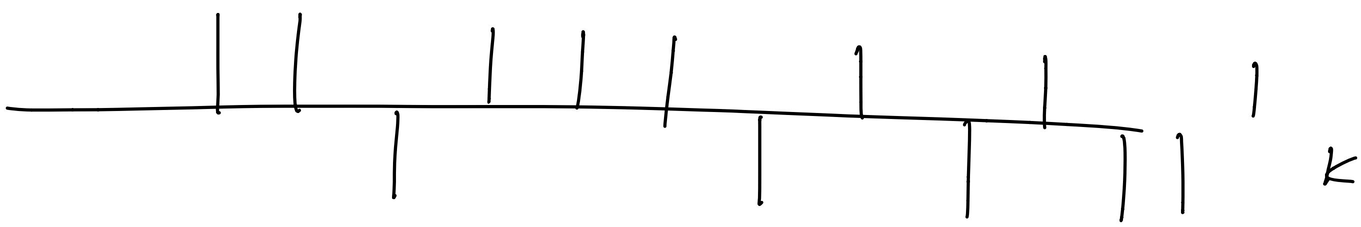
Some definitions through a discrete-time example

Bernoulli Process (discrete time, discrete valued)

X_k has independent samples.

Each is either +1 or −1.

$P(X_k = +1) = p \qquad P(X_k = -1) = 1-p.$



if $p = 1/2$, "binary white noise".

a wide sense Stationary process : The probability of a sequence with n +1's and m −1's is always $p^n(1-p)^m$ regardless of the starting point. (we will discuss this property more later)

Ergodic: $E(X_k) = 2p-1$, $Var(X_k) = 4p(1-p)$ } the same whether the average is taken across time or samples / realizations

Ensemble average $\mu_x(t) = E(x(t))$ ← average across samples

Time average $\hat{\mu}_x = \frac{1}{T}\int_0^T x(t)\,dt$ ← average across time

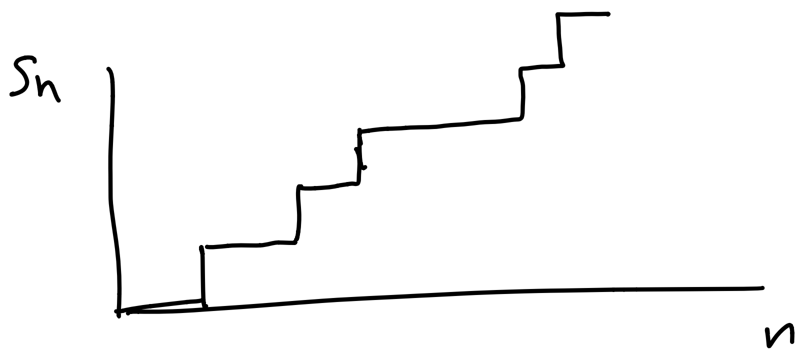
Binomial Counting Process

Let X_i be a sequence of i.i.d. Bernoulli RVs. (parameter p)

Let $S_n = \sum_{i=1}^{n} X_i$ (the sum of the 1st n trials)

S_n is a non decreasing function that grows by one @ random discrete times

Sample realization



At any time n, S_n (the RV) is a binomial RV with parameters (n, p).

So we can write its PMF:

$$p_{S_n}(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

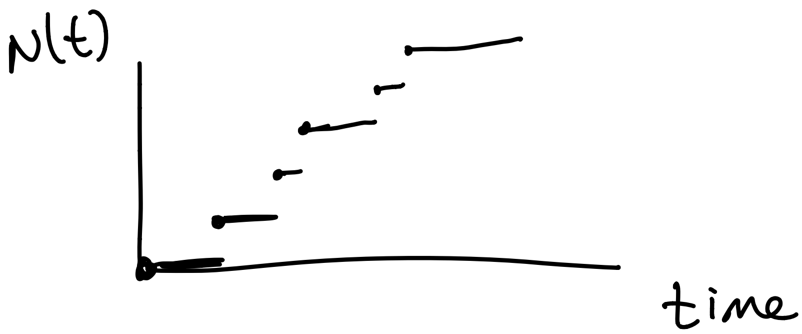
for $n > 0$ and $0 \le x \le n$

Poisson Process (continuous time, discrete valued)

$N(t)$ is the number of event occurences in a time interval $[0, t]$, where events occur at random at average rate λ

This is a non decreasing, integer-valued, continuous-time RP.

(Chapter 9.4)



We can view this as a limiting case of the binomial counting process.

Consider each time instance in the binomial process to be associated with a small continuous-time increment with duration δ.

- Only one event can occur inside that interval (because δ is so small)
  $\Rightarrow$ equivalent to a Bernoulli trial inside interval
- whether an event occurs in one interval is independent of it happening in any other

If the prob. an event occurs in a small interval δ is p, and $t = n\delta$, then the expected # occurences in time t is np.

The rate of occurences is λ per second, so $\lambda t = np$

If we let $n \to \infty$, keeping $t = \delta/n$,
and let $p \to 0$, keeping $np = \lambda t$ fixed, then
$P(N(t) = k) =$ poisson pmf. w/ $\alpha = \lambda t$

Poisson process w/ rate λ $N(t)$

- # arrivals in any interval (t_0, t_1) is a Poisson RV with expected value $\lambda(t_1 - t_0)$

  (Note: this depends on the length of the interval and is consistent w/ what we saw before.)

- For any non-overlapping interval, the # arrivals in each interval (which are both RVs) are independent

- $E(N(t))/t = \lambda$

If $N(t)$ is Poisson process w/ rate λ

1. # arrivals in any interval (t_0, t_1) is a Poisson RV with expected value $\lambda(t_1 - t_0)$

   (Note: this depends on the length of the interval and is consistent w/ what we saw before.)

why?

If the prob. an event occurs in a small interval δ is p, and $t = n\delta$, then the expected # occurences in time t is np.

The rate of occurences is λ per second, so $\lambda t = np$

If we let $n \to \infty$, keeping $t = \delta/n$, and let $p \to 0$, keeping $np = \lambda t$ fixed, then $P(N(t) = k)$ = poisson pmf. w/ $\alpha = \lambda t$

and $E(N(t)) = \lambda t$

2. The # occurences in any 2 non overlapping time intervals are independent RVs

3

Consider now the time interval T between any two events occuring in a Poisson process, example, between the $(n-1)^{th}$ and the n^{th} occurence

$$f_T(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0$$

exponential RV!

why?

$$F_T(t) = 1 - P(T > t)$$

$P(T > t) = P(\text{no events happen in } t \text{ seconds})$

Divide the interval $[0, t]$ into n small not overlapping subintervals, each length $\delta = t/n$

These are small enough the probability of 2 occurences happening in one small subinterval is basically 0.

Using our assumption earlier, $P(\text{occurence in small subinterval}) = p$,

then

$P(\text{no occurences in interval length } t = n\delta) = (1-p)^n$

which as $n \to \infty$ w/ $p = \frac{\lambda t}{n}$, this $\to e^{-\lambda t}$.

Discrete random walk (discrete time, discrete value)

$$X_k = X_0 + \sum_{l=1}^{k} B_l \qquad k > 0$$

X_0 initial condition

B_k a Bernoulli RP (recall: B_k are iid Bernoulli RVs)

($B_l = 0$ or $B_l = 1$ for all l)

$$E(X_k) = E(X_0) + \sum_{l=1}^{k} E(B_l)$$

$$= E(X_0) + k(2p-1)$$

The mean depends on time index k, so this is not a stationary process

Gaussian Process

$X(t)$ is a Gaussian RP
if and only if

$$\{X(t_1), X(t_2), \dots, X(t_n)\}$$

is a Gaussian vector for any n
and any $\{t_1, t_2, \dots, t_n\}$

(a similar definition for a discrete-time Gaussian Process)

Discrete-time system

Both X_k and Y_k are discrete-time RPs

$$Y_k = \sum_{l=0}^{n} a_l X_{k-l}$$