Functions of 2 RVs (Chapter 5.8)

If $Z = g(X, Y)$ what is $f_Z(z)$ in terms of $f_{XY}(x, y)$?

In general, use the 2-step process.

① Find $F_Z(z) = P(Z \le z)$, expressed in terms of $f_{XY}(x, y)$ or $F_{XY}(x, y)$

② differentiate to get $f_Z(z) = \frac{d}{dz} F_Z(z)$

Example functions

X is the signal amplitude @ a transmitter
Y is the attenuation factor between transmitter and receiver
$Z = X/Y$ is the signal amplitude at receiver

X and Y are two versions of the same signal arriving to a receiver by different paths.

- Selection diversity combining $Z = \max(|X|, |Y|)$
- equal gain combining $Z = X + Y$
- max-ratio combining $Z = aX + bY$ and choose a and b to optimize receiver

Case 1 $\quad Z = \max(X, Y)$ and X and Y are independent

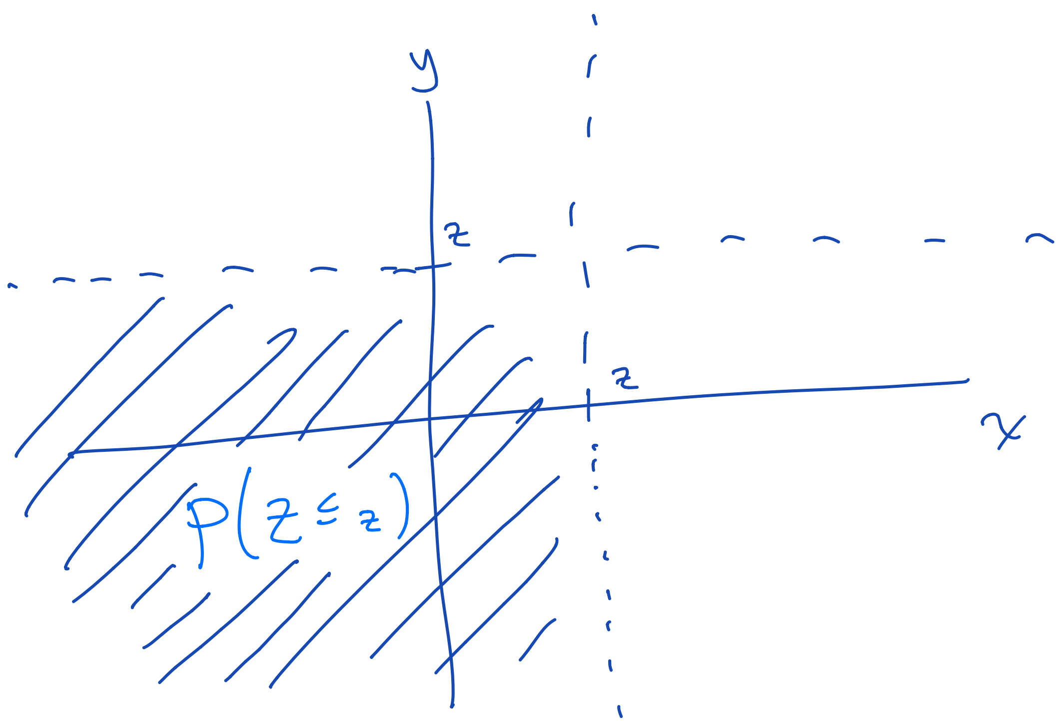
what is $F_Z(z)$?

$$F_Z(z) = P(Z \le z) = P(X \le z \text{ and } Y \le z)$$
$$= P(X \le z)\, P(Y \le z) = F_X(z)\, F_Y(z)$$

Can be extended to n independent RVs.

$Z = \max(X_1, X_2, \ldots, X_n)$ then

$$F_Z(z) = \prod_{i=1}^{n} F_{X_i}(z)$$



Case (2) $Z = \min(X, Y)$ and X and Y are independent

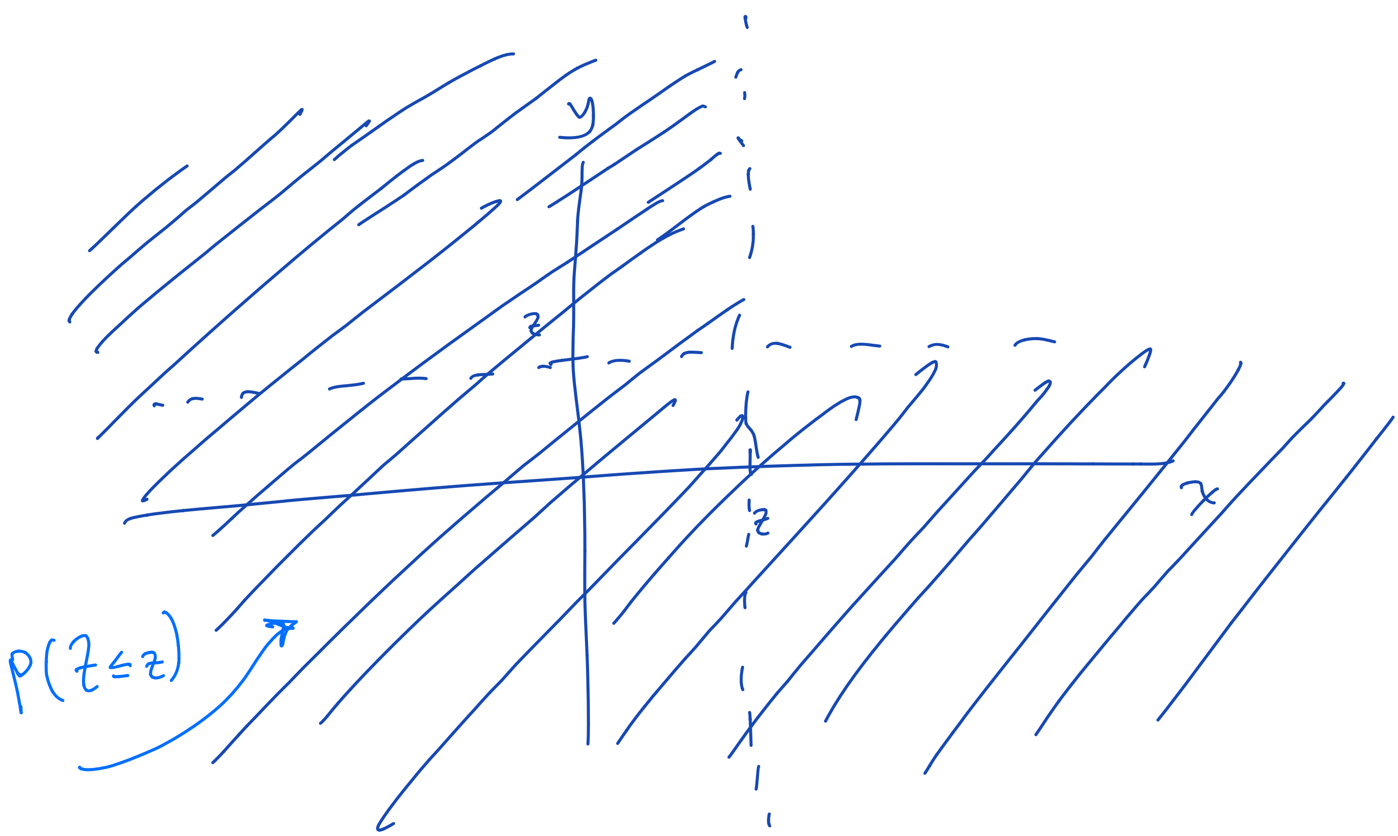
what is $F_Z(z)$?

$$F_Z(z) = P(Z \le z) = P(X \le z \text{ or } Y \le z)$$
$$= 1 - P(X > z \text{ and } Y > z)$$
$$= 1 - P(X > z)\,P(Y > z)$$
$$= 1 - (1 - F_X(z))(1 - F_Y(z))$$

Can be extended to n independent RVs

$Z = \min(X_1, X_2, \ldots, X_n)$ then

$$F_Z(z) = 1 - \prod_{i=1}^{n} (1 - F_{X_i}(z))$$



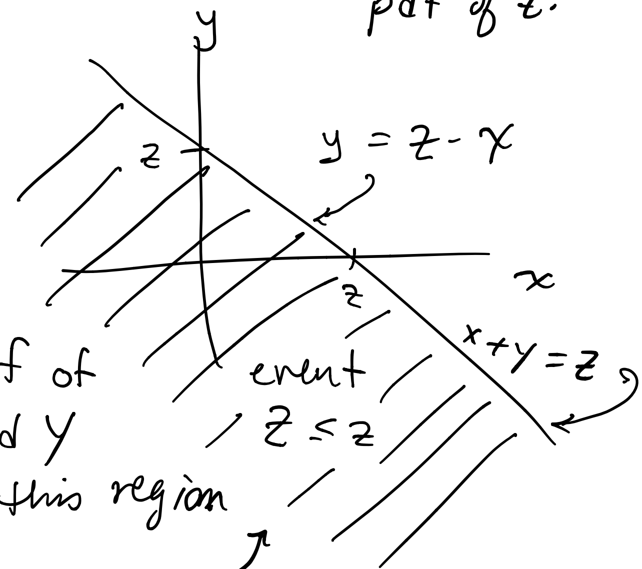
Sum of any 2 random variables (Chapter 5.8)

$Z = X + Y$. what is the pdf of Z?

Use the 2-step process. That is, find CDF of Z, then pdf of Z.

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$



so integrate joint pdf of X and Y over this region

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{XY}(x', y')\, dy'\, dx'$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x')\, dx'$$

true for any X and Y

Now suppose X and Y are independent

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x')\, f_Y(z - x')\, dx'$$

Convolution!

The pdf of the sum of 2 independent RVs is a convolution of their marginal pdfs

Example X and Y are independent, each is uniformly distributed between $[0,1]$. What is the PDF of $X+Y=Z$?

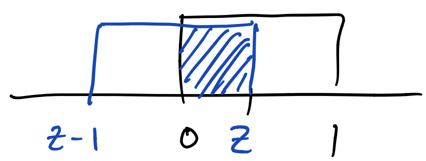
$$f_Z = f_X * f_Y \qquad f_X(x) = u(x) - u(x-1)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

Break into regions for different values of z based on overlap

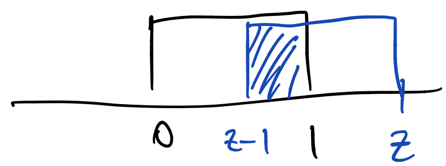
If $z<0$ or $z>2$, there's no overlap $\Rightarrow f_Z(z)=0$

If $0<z<1$



$$f_Z(z) = \int_0^z dx = z$$

if $1<z<2$



$$f_Z(z) = \int_{z-1}^{1} dx = 2-z$$

So
$$f_Z(z) = \begin{cases} 0 & z<0 \text{ or } z>2 \\ z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \end{cases}$$