A series of examples for estimating/bounding a probability when the PDF is unknown.

The orders at a restaurant are IID.
Estimate the probability the 1st 100 customers, combined, spend at least \$840.

In general
$$M_{100} = \frac{1}{100} \sum_{i=1}^{100} X_i$$

$$E(M_{100}) = E(X_i)$$
$$Var(M_{100}) = \frac{Var(X)}{100}$$

and
$$S_{100} = 100 M_{100} = \sum_{i=1}^{100} X_i$$

Case 1: $E(X) = \$8$ (and that's all we know)

use Markov inequality

$E(M_{100}) = 8$

$$P(S_{100} \geq 840) = P(M_{100} \geq 8.4) \leq \frac{E(M_{100})}{8.4}$$
$$= \frac{80}{84}$$

(awfully close to 1 to provide much info, but at least it's less than 1!)

Case 2: $E(X) = \$8$ and $STD(X) = \$2$

Use Chebyshev inequality

$$E(M_{100}) = 8 \quad \text{and} \quad Var(M_{100}) = \frac{(2)^2}{100} = 0.04$$

$$P(S_{100} \geq 840) = P(M_{100} \geq 8.4) = P(M_{100} - 8 \geq 0.4)$$

$$\leq P(|M_{100} - 8| \geq 0.4) \quad \text{(because it's a larger event)}$$

$$\leq \frac{Var(M_{100})}{(0.4)^2} = \frac{0.04}{(0.4)^2} = 0.25$$

(A much more useful upper bound)

Case 3: $E(X) = \$8$, $STD(X) = \$2$, and

M_{100} is "approximately Gaussian"

Use Central Limit Theorem

$$P(M_{100} \geq 8.4) = P\left(\frac{M_{100} - E(M_{100})}{\sqrt{var(M_{100})}} \geq \frac{8.4 - E(M_{100})}{\sqrt{VAR(M_{100})}}\right)$$

$$= P\left(Z \geq \frac{8.4 - 8}{\sqrt{0.04}}\right) = P\left(Z \geq \frac{0.4}{0.2}\right)$$

$$= 1 - P(Z < 2) = 1 - \Phi(2)$$

$$= 0.0228$$