Conditional Expectation (Chapter 5.7.2)

(a) Given an observed value of x, what is $E(Y|x)$?

$$\boxed{E(Y|x) = \int_{-\infty}^{\infty} y f_Y(y|x)\,dy}$$

this answer depends on x

Could just as well say

$E(Y|x) = g(x)$ for some x

(b) What is $E(Y|X)$? $= g(X)$

And $g(X)$ is a random variable, since X is a random variable

So we can take its expected value too.

$$E(E(Y|X)) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y f_Y(y|x)\,dy\right] f_X(x)\,dx$$

expectation wrt x

expectation wrt y

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} y f_Y(y)\,dy = E(Y)$$

So $E(Y) = E(E(Y|X))$

$= E_X(E_Y(Y|X))$ Law of iterated expectations

Also: $E(h(Y)) = E_X(E_Y(h(Y)|X))$

ex: $E(Y^k) = E(E(Y^k|X))$

This is extremely powerful!

The "traditional" approach to compute $E(Y)$:

$$\left.\begin{matrix} f_Y(y|x) \\ f_X(x) \end{matrix}\right\} \xrightarrow[\text{multiply}]{} f_{XY}(x,y) \xrightarrow[\text{integrate}]{} f_Y(y) \xrightarrow[\text{integrate}]{} E(Y)$$

Using iterated expectations to compute $E(Y)$:

$$f_Y(y|x) \xrightarrow{\text{integrate}} E(Y|X)$$

$$f_X(x) \xrightarrow{\text{integrate}} E(Y)$$

IF Y conditioned on X is a common RV like, say, exponential, then the top integration becomes trivial.

Example of why iterated expectations are so powerful.

Let X be uniform on $[0,1]$

Let Y be uniform on $[0,X]$

What is $E(Y)$?

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method 1 (the slow way)

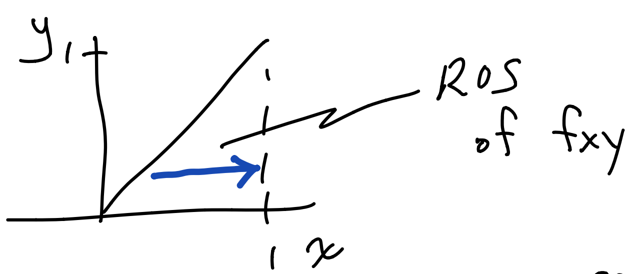
$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y|x) = \begin{cases} 1/x & 0 \le y < x \\ 0 & \text{else} \end{cases}$$

from the problem statement

joint pdf

$$f_{XY}(x,y) = f_Y(y|x) f_X(x) = \begin{cases} 1/x & 0 < y \le x \le 1 \\ 0 & \text{else} \end{cases}$$



marginal $\displaystyle f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 \frac{1}{x}\,dx$

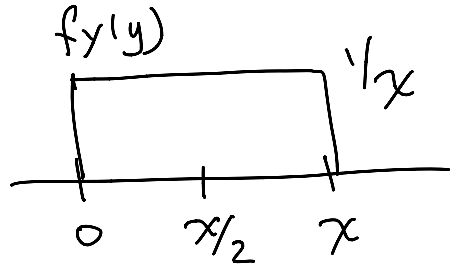
$$= \ln x \Big|_y^1 = -\ln y \quad \text{for } 0 < y < 1$$

expectation $\displaystyle E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = -\int_0^1 y \ln y\,dy = \ldots\ldots$

$= 1/4$ (details omitted)

Method 2 using iterated expectation

$$E(Y \mid X) = \frac{X}{2}$$



because the mean of a uniform RV on $[0, X]$ is $X/2$

$$E\left(E(Y \mid X)\right) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X)$$

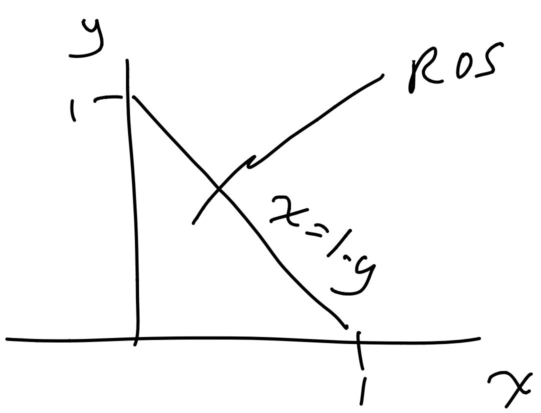
$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}$$

because the mean of X is $1/2$ because it's a uniform RV on $[0, 1]$

Another example of iterated expectations

X and Y are uniformly distributed on triangle formed by $(0,0), (1,0), (0,1)$

what is $E(X)$?



Steps:

a) Find $f_X(x|y)$

b) Compute $E(X|y)$

c) Compute $E(X) = E(E(X|Y))$

a) use chop and scale

For a given y, X is uniform on $[0, 1-y]$.

$$f_X(x|y) = \begin{cases} \frac{1}{1-y} & 0 \le x \le 1,\ x+y \le 1 \\ 0 & \text{else} \end{cases}$$

b) $E(X|y) = \frac{1-y}{2}$ because $X|y$ is uniform

c) $E(X) = E\left(\frac{1-Y}{2}\right) = \frac{1}{2} - \frac{E(Y)}{2}$

But by symmetry, we can also say that

$$E(X) = E(Y)$$

Combining, $E(Y) = \frac{1}{2} - \frac{E(Y)}{2} \Rightarrow E(Y) = \boxed{E(X) = \frac{1}{3}}$