Independence of RVs (Chapter 5.5)

Recall: independence of events:

Events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

Extend to 2 RVs, either through PMF or CDF.

$$\boxed{\text{Random variables } X \text{ and } Y \text{ are independent if and only if } F_{XY}(x,y) = F_X(x)F_Y(y) \text{ for all } x \text{ and } y.}$$

Recall that $F_{XY}(x,y) = P(X \le x \text{ and } Y \le y)$

and $F_X(x)F_Y(y) = P(X \le x)P(Y \le y)$

so this is really saying that the events $\{X \le x\}$ and $\{Y \le y\}$ are independent for any x and any y.

From this, it is easy to see that independence occurs if

$$p_{XY}(x,y) = p_X(x)\,p_Y(y) \quad \text{for all } x,y$$

and

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) \quad \text{for all } x,y$$

<u>and</u> that either of these conditions leads to independence

what good is independence?

Simplifying your model

<u>If</u> you are given $f_X(x)$ and $f_Y(y)$

<u>and</u> X and Y are independent

then simplify $f_{XY}(x,y) = f_X(x)\, f_Y(y)$.

If you are given $f_{XY}(x,y)$ and

asked if X and Y are independent,

then you need to show

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \quad \text{for all } x \text{ and } y.$$

One counter-example is sufficient to show the lack of independence

Recall: Given $f_{XY}(x,y)$, compute marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx.$$

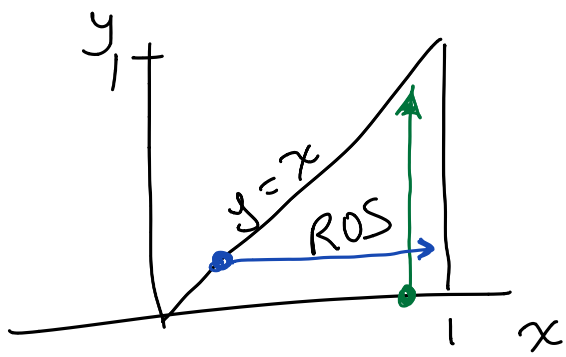
Independence is the ONLY situation in which, given $f_X(x)$ and $f_Y(y)$, you can recover the joint $f_{XY}(x,y)$ (marginals do not tell the full story otherwise)

Example (same as before)

Two random variables are uniformly distributed over the triangle formed by $(0,0)$, $(1,1)$, $(1,0)$.

Are they independent?

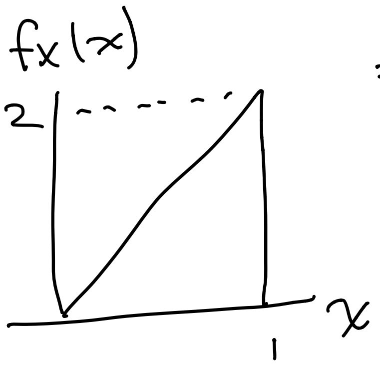
$$f_{xy}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{else} \end{cases}$$



view from top-down, view the x-y plane

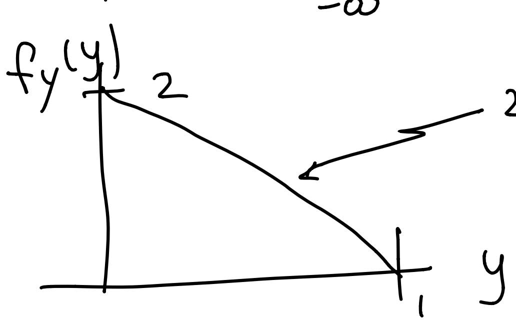
$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

for a given x inside the ROS, y varies from 0 to x



$$= \int_0^x 2\,dy = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_y^1 2\,dx = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$



for a given y inside the ROS, x varies from y to 1

Are X and Y independent?

does $f_{XY}(x, y) = f_X(x) f_Y(y)$ for all x and y?

$$f_X(x) f_Y(y) = (2x)\,2(1-y) \quad \text{for } 0 \le x \le 1,\ 0 \le y \le 1$$

$$\neq 2 \quad \text{for } 0 \le y \le x \le 1$$

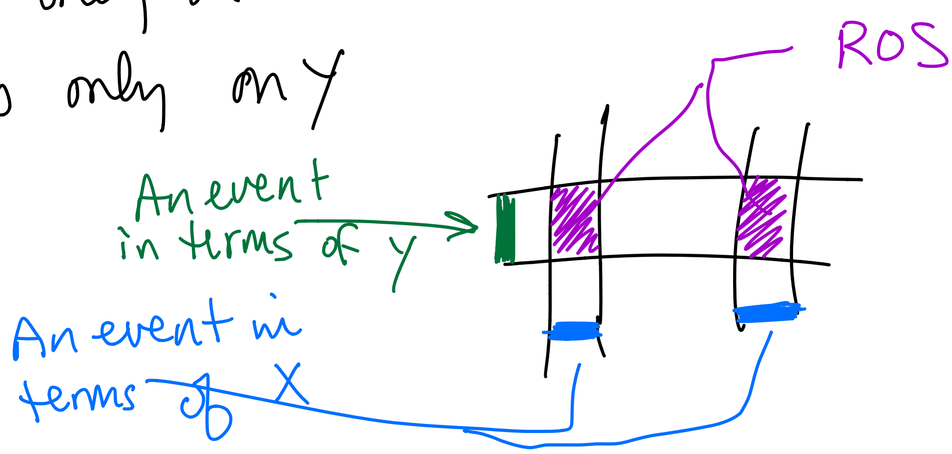
so **No**, not independent

Note: the ROS's must also match, which they don't here. One is triangular, the other square.

> If $f_{XY}(x, y)$ has a ROS that is not in "product form", then X and Y cannot be independent

product form: A region can be written as the intersection of an event A that depends only on X and an event B that depends only on Y

An example of product form



Some more examples

$$f_{xy}(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

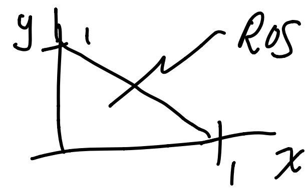
$$f_x(x) = \int_0^1 4xy\, dy = \frac{4xy^2}{2}\Big|_0^1 = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_y(y) = \int_0^1 4xy\, dx = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_x(x) f_y(y) = (2x)(2y) \quad \text{when } 0 \le x \le 1 \text{ and } 0 \le y \le 1$$

$$= f_{xy}(x,y) \Rightarrow \underline{\text{YES}} \text{ independent}$$

$$f_{xy}(x,y) = \begin{cases} 24xy & 0 \le x,\ 0 \le y,\ x+y \le 1 \\ 0 & \text{else} \end{cases}$$



$$f_x(x) = \int_0^{1-x} 24xy\, dy = \begin{cases} 12x(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

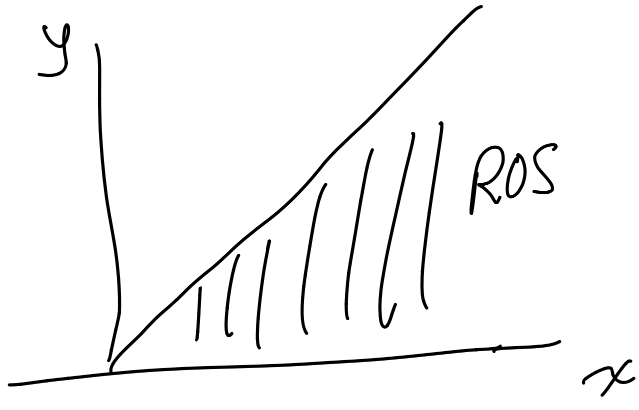
$$f_y(y) = \int_0^{1-y} 24xy\, dx = \begin{cases} 12y(1-y)^2 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

and clearly $f_{xy}(x,y) \ne f_x(x) f_y(y)$ $\boxed{\text{NOT}}$

→ the triangular ROS changes everything

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{else} \end{cases}$$

ROS is triangular, so NOT independent.



We can also compute

$f_X(x) = 2e^{-x}(1-e^{-x}) \quad x > 0$

$f_Y(y) = 2e^{-2y} \quad y > 0$

$\Rightarrow$ NOT independent

A CDF example

$$F_{XY}(x,y) = \begin{cases} (1-e^{-ax})(1-e^{-by}) & x \ge 0,\ y \ge 0 \\ 0 & \text{else} \end{cases}$$

YES Independent, since

$F_X(x) = \lim_{y \to \infty} F_{XY}(x,y) = 1 - e^{-ax} \quad x \ge 0$

$F_Y(y) = \lim_{x \to \infty} F_{XY}(x,y) = 1 - e^{-by} \quad y \ge 0$

$\boxed{\text{if } X \text{ and } Y \text{ are independent, then } g_1(X) \text{ and } g_2(Y) \text{ are also independent}}$