Example with exponential RV.

Let T be the duration of a telephone call, which is modeled by an exponential RV with $\lambda = 1/3$. (This means $E(T) = 1/\lambda = 3$)

What is the conditional pdf of the duration, for calls that last more than 2 minutes?

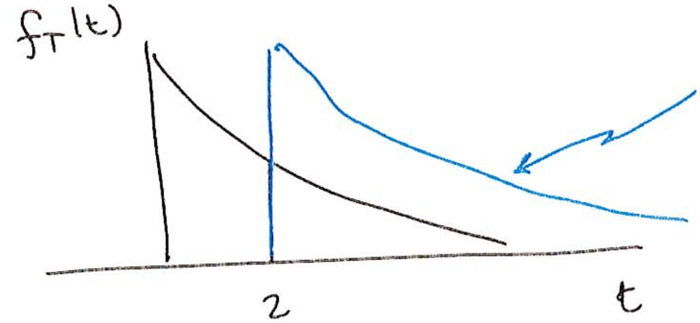
Answer: Conditioning event $\{T > 2\}$

Recall $f_T(t) = \begin{cases} 1/3\, e^{-t/3} & t > 0 \\ 0 & \text{else} \end{cases}$

$$P(T > 2) = \int_2^\infty \frac{1}{3} e^{-t/3}\, dt = e^{-2/3}$$

Conditional pdf $f_T(t \mid T > 2)$

$$= \begin{cases} \dfrac{f_T(t)}{P(T>2)} & t > 2 \\ 0 & \text{else} \end{cases} = \begin{cases} \dfrac{\frac{1}{3} e^{-t/3}}{e^{-2/3}} = \frac{1}{3} e^{-(t-2)/3} \\ 0 \end{cases}$$



$f_T(t \mid T > 2)$ – a time shifted exponential!

The exponential RV is memoryless

memoryless:

$$\underbrace{P(X > t+h \mid X > t)}_{\text{wait at least } h \text{ more seconds given already waited } t \text{ seconds}} = \underbrace{P(X > h)}_{\text{wait at least } h \text{ seconds at the beginning of the wait}} \quad \text{for } h > 0$$

Show by 1st principles

$$P(X > t+h \mid X > t) = \frac{P(\{X > t+h\} \cap \{X > t\})}{P(X > t)}$$

$$= \frac{P(X > t+h)}{P(X > t)}$$

because clearly if $X > t+h$, for $h > 0$, then $X > t$ also

$$= \frac{1 - F_X(t+h)}{1 - F_X(t)}$$

$$= \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = P(X > h)$$

Recall

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$$

The system "forgets" how long you've already waited

$\Rightarrow$ "memoryless"

or, it doesn't matter where you start counting/observing or where you set $t = 0$

Geometric RV is also memoryless!

Conditional Expected Value

Given RV X and conditioning event B,

$$E(X|B) = m_{X|B} = \sum_{x \in S_X} x\, p(x|B)$$

$$= \int_{-\infty}^{\infty} x\, f_X(x|B)$$

(absolute convergence is required)

Conditional mean is the mean of the conditional pdf

conditional variance

Computed using conditional mean

$$\mathrm{Var}(X|B) = E\left[(X - m_{X|B})^2 \,|\, B \right]$$

$$= E(X^2|B) - E(X|B)^2$$

Conditional variance is the variance of the conditional pdf

Theorem of total expectation

$$E(X) = \sum_{i=1}^{n} E(X|B_i)\, P(B_i)$$ if B_i's form a partition

and $$E(g(X)) = \sum_{i=1}^{n} E(g(X)|B_i)\, P(B_i)$$

follows directly from

$$f_X(x) = \sum_{i=1}^{n} f_X(x|B_i)\, P(B_i)$$

Example conditional expected value

Duration T, exponential $\lambda = 1/3$.

For calls that last at least 2 minutes, what is the expected length of the call?

Know from before that

$$f_T(t \mid T>2) = \begin{cases} \frac{1}{3} e^{-(t-2)/3} & t>2 \\ 0 & \text{else} \end{cases}$$

$$E(T \mid T>2) = \int_2^\infty t \, \frac{1}{3} e^{-(t-2)/3} \, dt$$

$$= -t \, e^{-(t-2)/3} \Big|_2^\infty + \int_2^\infty e^{-(t-2)/2} \, dt$$

$$= 2 + 3 = 5$$

Another math approach:

change of variable s, let $s = t-2$ so $t = s+2$

$$E(T \mid T>2) = \int_0^\infty (s+2) \frac{1}{3} e^{-s/3} \, ds$$

$$= \underbrace{\int_0^\infty s \, \frac{1}{3} e^{-s/3} \, ds}_{E(T)} + \underbrace{2 \int_0^\infty \frac{1}{3} e^{-s/3} \, ds}_{2(1)}$$

$$= E(T) + 2 = 3 + 2 = 5$$

Example (lifetime)

Suppose someone has already lived to be 70 years old

Let X be the additional number of years lived

This depends on whether they have high or regular blood pressure, which 40% of them do.

If $H = \{\text{has high blood pressure}\}$

$$P_X(x|H) = \begin{cases} 0.1\,(0.9)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

("a ten percent chance of dying each year")

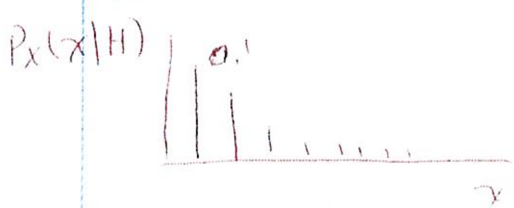
If not H

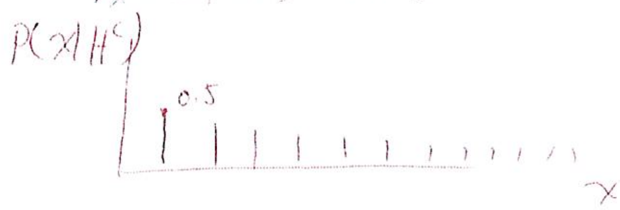
$$P_X(x|H^c) = \begin{cases} 0.05\,(0.95)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

("a 5% chance of dying each year")

"failure" (ie, no death) on 1st $x-1$ trials, "success" on x^{th} trial

$$P_X(x) = P_X(x|H)\,P(H) + P_X(x|H^c)\,P(H^c)$$





$$E(X) = E(X|H)\,P(H) + E(X|H^c)\,P(H^c)$$
$$= \left(\frac{1}{0.1}\right)(.4) + \left(\frac{1}{0.05}\right)(.6) = 16 \text{ years}$$

Example from previous exam

Problem 46. (MULTIPLE CHOICE: 5 POINTS)
Three engineers, Jan, Pat, and Rory, are processing work orders. The time it takes each to finish one work order is an exponential random variable. Jan takes an average of 3 hours; Pat takes an average of 1 hours, and Rory takes an average of 4 hours. Because of their speed, Pat processes 50% of the work orders, while Jan and Rory each process 25% of them.
What is the mean time (in hours) it takes any given work order to be completed?

(a) 2

(b) 9/4

(c) 8/3

(d) 8

(e) None of the above

(f) Too little information to solve.

Problem 47. (16 POINTS)
A customer walks into a store and is equally likely to be served by one of three clerks. The time taken by the first clerk is an exponetial RV with mean 2; the time taken by the second clerk is an constant RV with mean 1; and the time taken by the third clerk is a uniform RV between zero and two.

(a) Express the PDF of T the time to serve the customer.

(b) Find $E(T)$.

T = time to complete a given work order.

T_J = time if Jan does it $\qquad T_J \sim Exp(1/3)$

T_P = time if Pat does it $\qquad T_P \sim Exp(1/1)$

T_R = time if Rory does it $\qquad T_R \sim Exp(1/4)$

$E(T|J) = E(T_J) = 3 \qquad E(T_P) = 1 \qquad E(T_R) = 4$

$P(P) = 1/2 \qquad P(J) = 1/4 \qquad P(R) = 1/4$

$$E(T) = E(T|J)P(J) + E(T|P)P(P) + E(T|R)P(R)$$

$$= 3(1/4) + 1(1/2) + 4(1/4)$$

$$= \frac{3}{4} + \frac{1}{2} + 1 = \boxed{\frac{9}{4}}$$

Harder approach

$$f_T(t) = f_T(t|J)P(J) + f_T(t|P)P(P) + f_T(t|R)P(R)$$

$$= \exp(-t/3)(1/4) + \exp(-t)(1) + \exp(-t/4)(1/4)$$

when $t > 0$

$$E(T) = \int_{-\infty}^{\infty} t f_T(t)\,dt$$

and then a lot of integration by parts

Ex. conditional mean

A mouse is placed @ center of a maze.
There are 3 ~~doors~~ paths.

Path 1. returns him to center after 2 min.
Path 2. returns him to center after 4 min
Path 3: allows him to exit after 1 min

Based on width of path

$P(\text{Path} 1) = 0.5$
$P(\text{Path} 2) = 0.3$
$P(\text{Path} 3) = 0.2$

What is expected time before he exits?

M = minutes before exit
D = number of path chosen

$$E(m \mid D=3) = 1$$
$$E(m \mid D=1) = 2 + E(m)$$
$$E(m \mid D=2) = 4 + E(m)$$

$$E(m) = E(m \mid D=1)\, P(D=1) + E(m \mid D=2)\, P(D=2) + E(m \mid D=3)\, P(D=3)$$

$$= 0.5\,[2 + E(m)] + 0.3\,[4 + E(m)] + 0.2\,(1)$$

an algebraic equation that can be solved $\boxed{E(M) = 12 \text{ min}}$