Poisson RV, parameter α (Chapter 3.5.4)

Counting # occurences in a given time period or a given spatial region, if events are "completely random" in space or time

Ex: # emissions from a radioactive substance

requests for telephone connections (arrivals)

defects in a chip

$$P(N=k) = p_N(k) = \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0, 1, 2, \ldots$$

$\alpha \equiv$ average # of events occuring in a specific interval or region

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pmf sums to 1: $\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$

$= e^{-\alpha} e^{\alpha} = 1$ ✓

$E(N) = \alpha$

$\text{var}(N) = \alpha$

Good for modeling when events are rare (small p) but the # trials is large (large n)

Approximates binomial well in these instances, and its much easier to compute than binomial w/ large n

Examples Poisson

number of queries, N, at a call center, in t seconds is a Poisson RV with $\alpha = \lambda t$ where λ is the average arrival rate in a period of time t.

Ex: $\lambda = 4$ queries/minute
(note: units of α are same as units of RV)

Q1: find the probability of more than 4 queries in 10 seconds.

Given λ, find α.

$$\alpha = \left(\frac{4 \text{ queries}}{\text{minute}}\right)\left(\frac{1 \text{ minute}}{60 \text{ seconds}}\right)\left(10 \text{ seconds}\right) = \frac{2}{3} \text{ queries}$$

(units of RV..)

$$P(N > 4) = 1 - P(N \leq 4)$$
$$= 1 - \sum_{k=0}^{4} \frac{(2/3)^k}{k!} \exp(-2/3)$$
$$\approx 6.33 * 10^{-4}$$

makes sense this is small, because if the avg # queries in 10 seconds is 2/3, it's quite unlikely to get more than 4..

Q2: What is the probability there are fewer than 6 queries in 2 minutes?

Given the new time duration, we have a new α.

$$\alpha = \left(4 \frac{\text{queries}}{\text{minute}}\right)\left(2 \text{ minutes}\right) = 8 \text{ queries}$$

$$P(N \leq 5) = \sum_{k=0}^{5} \frac{8^k}{k!} \exp(-8) \approx 0.1$$

This also makes sense – the average is 8, so it's unlikely there are 5 or less

Exercise 19.
The following physical scenario is best modeled by which type of random variable?
Each millisecond at a telephone switch, a call independently arrives with probability $p = 0.01$. Let T be the number of milliseconds until you receive 100 calls.

(a) Poisson

(b) Exponential

(c) Gaussian

(d) Uniform

Exercise 20.
The following physical scenario is best modeled by which type of random variable?
An LCD display has 1000*75 pixels. A display is accepted if it has 15 or fewer faulty pixels. The probability that a pixel is faulty is 1e-5. Find the proportion of displays that are accepted.

(a) Poisson

(b) Exponential

(c) Gaussian

(d) Uniform

Exercise 21.
The following physical scenario is best modeled by which type of random variable?
A quantizer is used to convert an analog signal (speech or audio) into digital form. A quantizer maps a random voltage X into the nearest point $q(X)$ from a set of representative values. The value X is then approximated by $q(X)$, which can be represented by an R-bit binary number. The quantization error $Z = X - q(X)$ varies between $-d/2$ and $d/2$, where d is the quantizer step size. What is a good model for Z when R is somewhat large?

(a) Poisson

(b) Exponential

(c) Gaussian

(d) Uniform

Another Poisson Example

Packets in a network arrive at a node at a rate of 100 packets per minute. What is the probability no packets arrive in 6 seconds? What is the probability 2 or more packets arrive in 6 seconds?

$$\alpha = \lambda t = \left(100 \frac{\text{packets}}{\text{minute}}\right)\left(\frac{6}{60} \text{ minutes}\right)$$

$$= 10 \text{ packets}$$

$$p_N(0) = \frac{\alpha^0}{0!} e^{-\alpha} = e^{-\alpha} = e^{-10} \approx 4.5 \times 10^{-4}$$

$$p(N \geq 2) = 1 - P(N=0) - P(N=1)$$

$$= 1 - p_N(0) - p_N(1)$$

$$= 1 - e^{-10} - \frac{\alpha}{1} e^{-10}$$

$$\approx 0.9995$$

Poisson approximates binomial for large n, small p.

Define $\alpha = np$, and let $n \to \infty$.

Then $p_N(k) = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{\alpha^k}{k!} e^{-\alpha}$

when $\alpha = np$

<u>Proof</u>: step 1: examine case where no event occurs in n trials.

$$p_N(0) = (1-p)^n = \left(1 - \frac{\alpha}{n}\right)^n \to e^{-\alpha} \text{ as } n \to \infty$$

Step 2: Consider the ratio $\frac{p_N(k+1)}{p_N(k)}$ for n trials.

$$\frac{p_N(k+1)}{p_N(k)} = \frac{n!}{(k+1)!\,(n-k-1)!} \quad \frac{k!\,(n-k)!}{n!} \quad \frac{p^{k+1}(1-p)^{n-k-1}}{p^k\,(1-p)^{n-k}}$$

$$= \frac{n-k}{k+1}\,\frac{p}{1-p} = \frac{np - pk}{(k+1)\left(1 - \alpha/n\right)} = \frac{\alpha\left(1 - \frac{k}{n}\right)}{(k+1)\left(1 - \alpha/n\right)}$$

This $\to \frac{\alpha}{k+1}$ as $n \to \infty$

So, as $n \to \infty$, $p_N(k+1) \approx \frac{\alpha}{k+1} p_N(k) = \frac{\alpha}{k+1}\,\frac{\alpha}{k}\,\frac{\alpha}{k-1} \cdots \frac{\alpha}{1}\, p_N(0)$

and so $p_N(k) = \frac{\alpha^k}{k!} e^{-\alpha}$ (since $p_N(0) = e^{-\alpha}$)

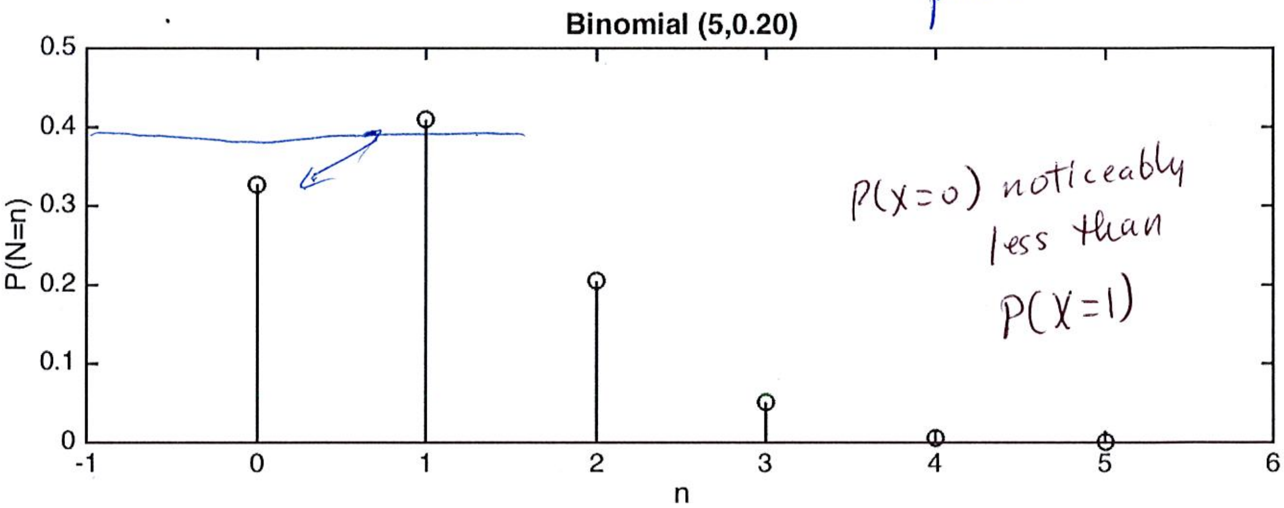
What is large n? How small is small p?

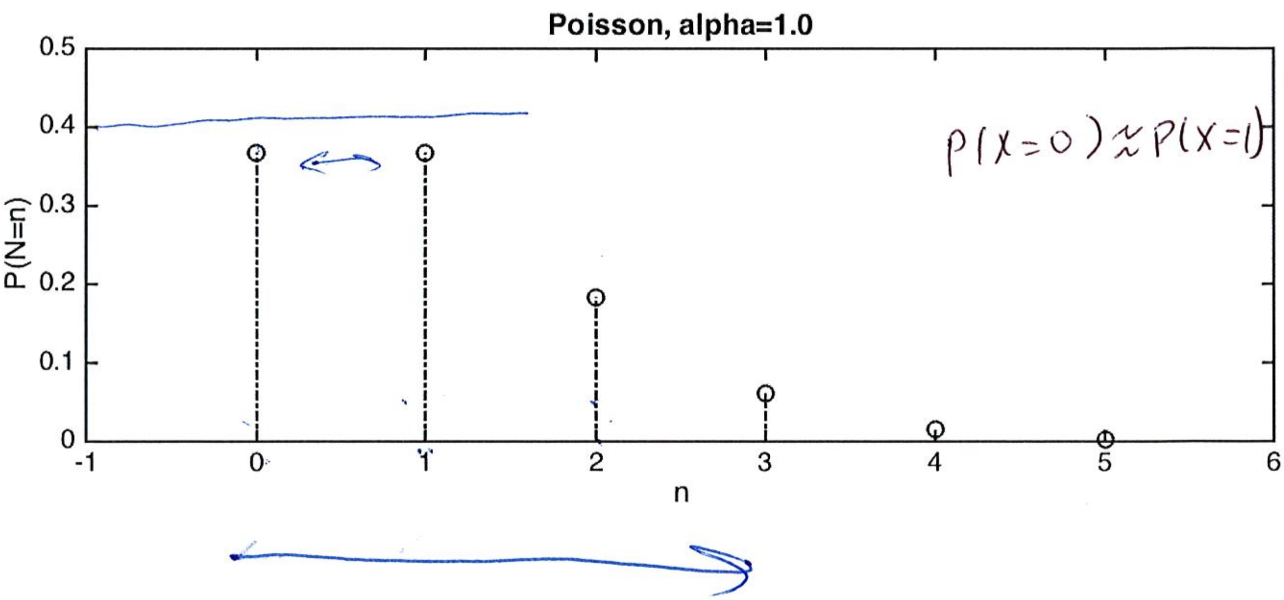
A rather small n

pmf of each RV

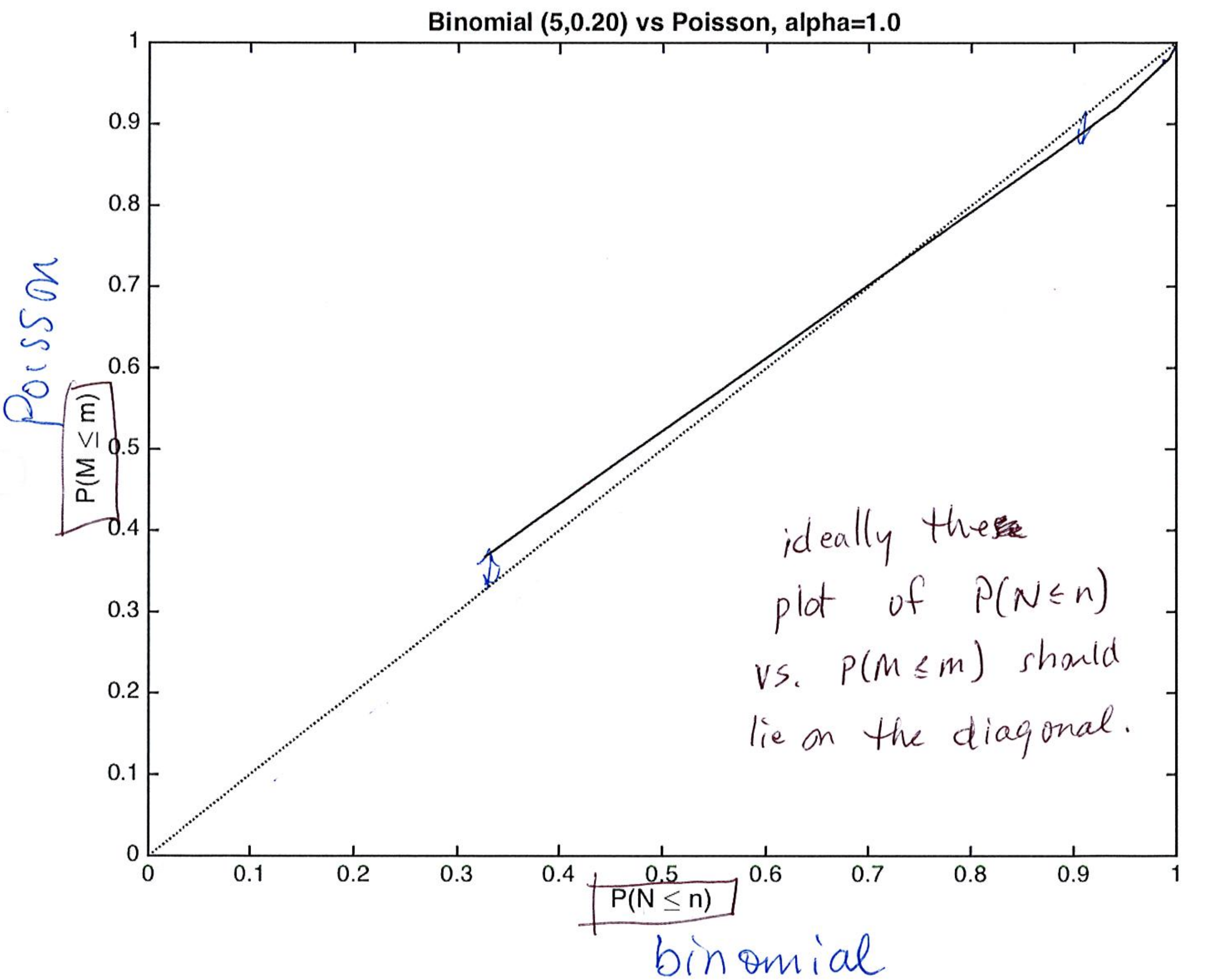
n = 5
p = .2

α = 1





n = 5, p = .2   α = 1

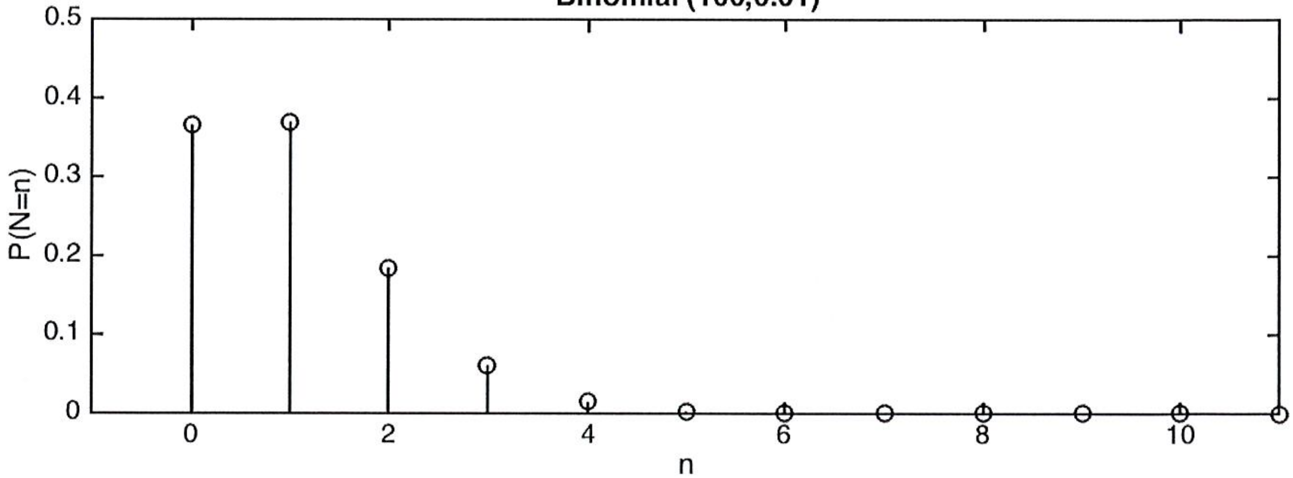


Binomial (5,0.20) vs Poisson, alpha=1.0
P(M ≤ m)
Poisson
P(N ≤ n)
binomial
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ideally the plot of P(N ≤ n) vs. P(M ≤ m) should lie on the diagonal.

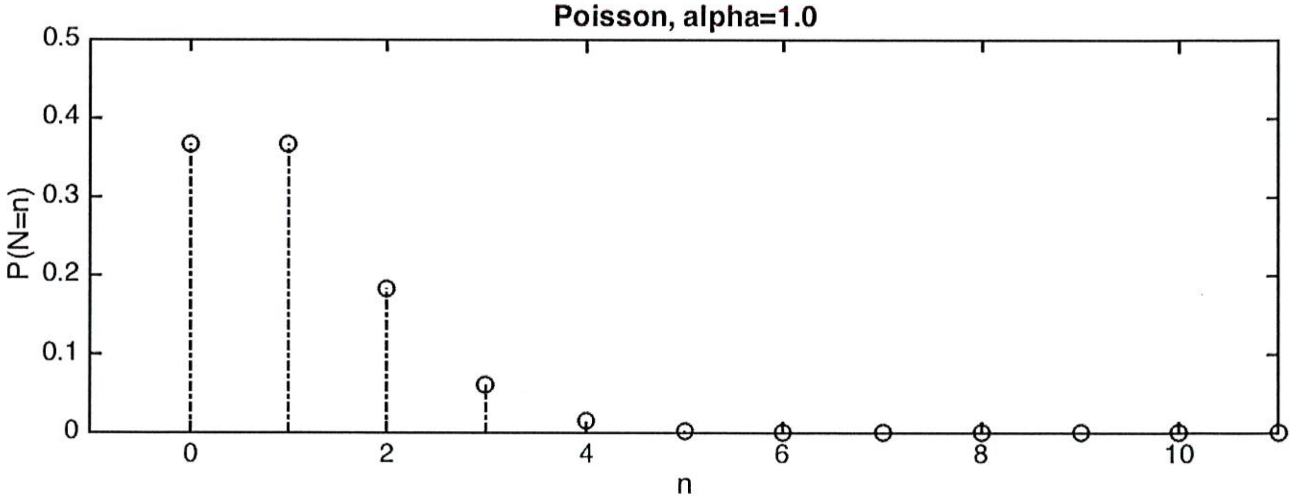
increase n, same α

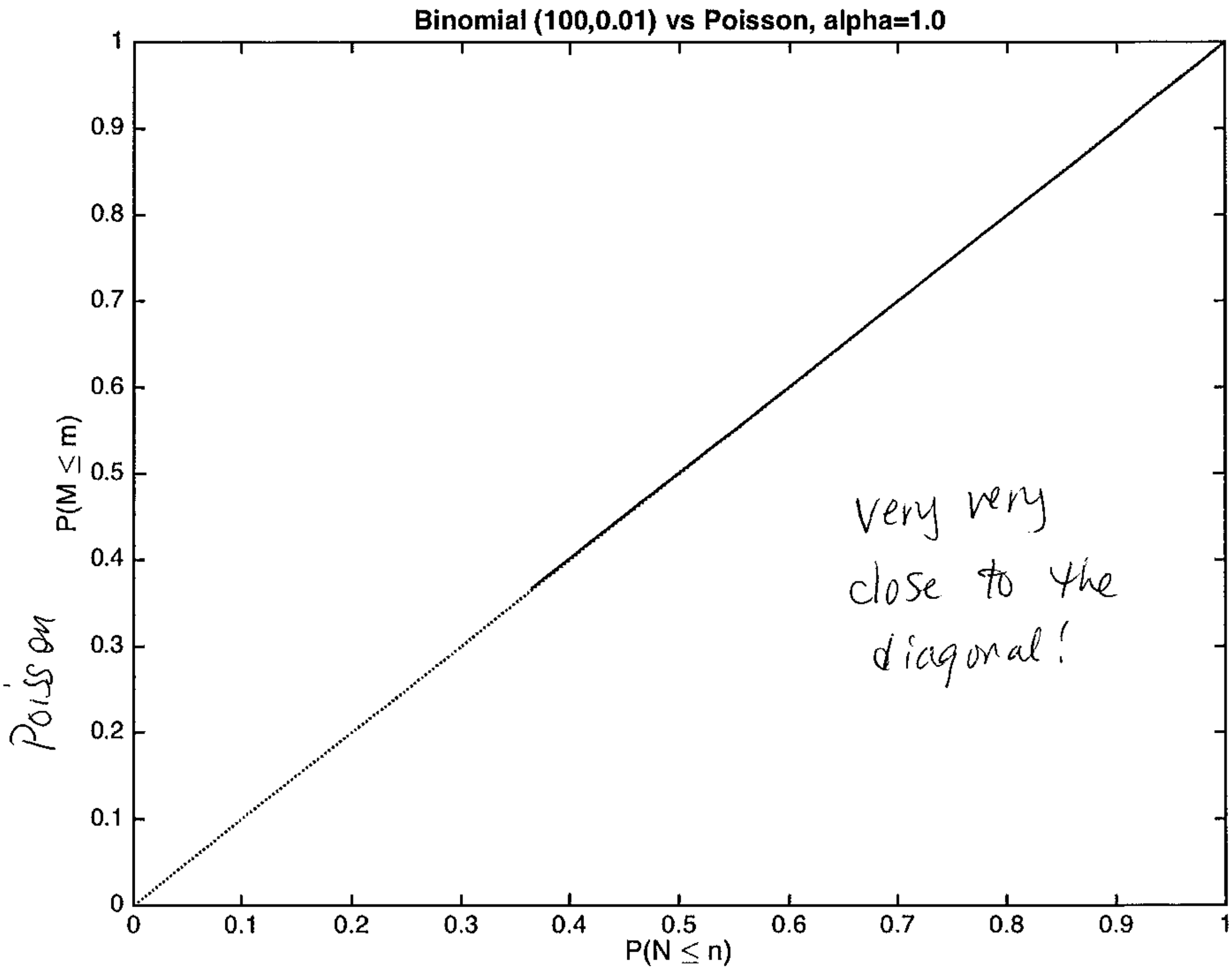
$n=100 \quad p = \frac{1}{100}$

Binomial (100,0.01)



Poisson, alpha=1.0





Binomial (100,0.01) vs Poisson, alpha=1.0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
P(M ≤ m)
Poisson
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P(N ≤ n)
very very
close to the
diagonal!

binomial