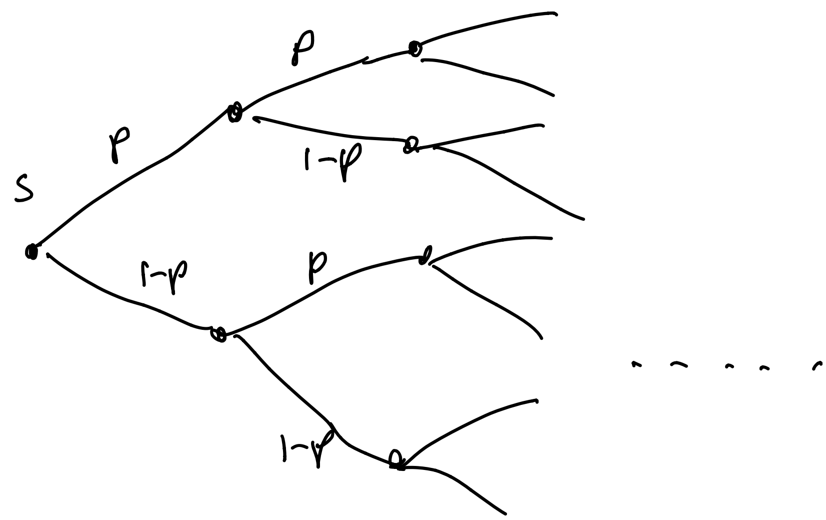
Binomial RV, parameters n and p (Section 2.6.2 and 3.5.2)

Repeat a Bernoulli trial/experiment n independent times

X = # times event A happens in n trials



Each branch of the tree has the same depth, because you perform the subexperiments regardless of the outcome of the previous stage

$S_X = \{0, 1, \dots, n\}$ (finite)

Pick a single leaf node with k successes what's the probability of reaching that node?

p is the probability of one success

p^k is the probability of k successes

$(1-p)^{n-k}$ is the probability of $n-k$ failures

$\Rightarrow p^k (1-p)^{n-k}$

And how many leaf nodes have exactly k success after n trials? $\binom{n}{k}$

Combined, $p_X(k) = P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}$

Why $\binom{n}{k}$ on previous page? Counting

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{symmetric: } \binom{n}{k} = \binom{n}{n-k}$$

A quick counting lesson (section 2.3.1 and 2.3.2)

2.3.1 Sampling with replacement with order

Choose k items from n possible items

- pay attention to the order when sampling
- we can have duplicates

Elements of the sample space are each an ordered k-tuple $(x_1, x_2, \ldots, x_k)$

Each element can be any of the n possible, so n^k possible elements in the sample space

2.3.2 Sampling without replacement with order

- pay attention to the order when sampling
- no duplicates allowed

Sample space contains k-tuples, but fewer.

Once the first item is picked, there's one fewer option

$$\text{so } |S| = \underbrace{n \cdot (n-1)(n-2) \cdots (n-k+1)}_{k \text{ numbers}}$$

$$= \frac{n!}{(n-k)!}$$

2.3.3 Permutations (how many ways can you order k items?)

Among your k items, you can order them $k(k-1)(k-2)\cdots 1 = k!$ different ways

2.3.4 Sampling without replacement without order

- no duplicates allowed
- only important which items, not when they're picked

K objects from a set of n distinct objects

A 2-step process:

① choose which k out of n items, paying attention to order (see 2.3.2)

$n!/(n-k)!$ possible ways

② divide by the number of times over counted (when we paid attention to order)

How many ways can you order k items? (see Sect 2.3.3)

$k!$ ways

Combining

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C_k^n$$

This is the binomial coefficient, stated "n choose k".

BTW: can approximate $n!$ using Stirling's formula

$$n! \approx \sqrt{2\pi}\; n^{n+1/2}\; e^{-n}$$

How does this relate to binomial RV?

Example 2.21 from section 2.3.4 (no replacement, no order)

In an urn, k white balls and $n-k$ black balls.
What is the number of distinct permutations?
(how many ways can white + black balls be ordered?)

This problem is equivalent to the following problem:
Put n tokens numbered 1 to n into a hat,
Pick K tokens to tell you the position
to put each white ball. The black balls fill in the gaps.

Each possible combination of K tokens creates
a distinct permutation of K white and $n-k$ black balls.

The number of permutations is $\binom{n}{K} = \frac{n!}{K!(n-k)!}$

Example: $n=5$ and $k=2$

Let "1" denote white ball and "0" denote black ball.

11000 01100 00110 00011
10100 01010 00101
10010 01001
10001

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\cdot4\cdot\cancel{3\cdot2\cdot1}}{2\cdot1\cdot\cancel{3\cdot2\cdot1}}$$

$$= \frac{5\cdot4}{2} = 10 \text{ possibilities}$$

Example : Flip a (not necessarily fair) coin 3 times.
what is the probability of k heads?

let $p = P(H)$
$1-p = P(T)$

sequence of flips	probability	# heads
H H H	$p \cdot p \cdot p$	3
H H T	$p \cdot p(1-p)$	2
H T H	$p(1-p)\, p$	2
H T T	$p(1-p)^2$	1
T H H	$(1-p)\, p^2$	2
T H T	$(1-p)^2 p$	1
T T H	$(1-p)^2 p$	1
T T T	$(1-p)^3$	0

X = # heads

Binomial: $p_X(k) = \binom{3}{k} p^k (1-p)^{n-k}$

x	$p_X(x)$	
0	$(1-p)^3$	$= \binom{3}{0} p^0 (1-p)^3$
1	$3p(1-p)$	$= \binom{3}{1} p^1 (1-p)^2$
2	$3p^2(1-p)$	$= \binom{3}{2} p^2 (1-p)^1$
3	p^3	$= \binom{3}{3} p^3 (1-p)^0$

Two applications of binomial RVs

1) Resource allocation
2) Error correcting codes

Resource allocation: You want to buy "resources" to handle customers, but you know not all your customers will need the resources at once.
How many resources should you buy to make sure the probability there aren't enough resources is less than some value?

Example 2.39 Packet voice

You decide to admit up to $n = 8$ customers at a time, knowing each will only be speaking for a fraction of the time. But you only have enough bandwidth to actually send voice from 6 speakers at a time. If each speaker is active with probability $p = 1/3$, what is the probability there are more than 6 speakers (ie, what is the probability of overload)

Answer: 8 independent Bernoulli trials

N = # active speakers of the 8

$$p_N(k) = \binom{8}{k} p^k (1-p)^{8-k} \qquad \text{where } p = 1/3$$

Want $P(N > 6) = P(N=7) + P(N=8)$

$$= \binom{8}{7} p^7 (1-p) + \binom{8}{8} p^8$$

$$= 0.00259 \Rightarrow \text{less than } 1\% \text{ of time}$$

Related question

what value of n will maximize your revenue if the probability of overload is less than 1% ?

$$P(\text{overload}) = P(N > 6) < 0.01$$

$$\sum_{k=7}^{n} \binom{n}{k} p^k (1-p)^{n-k} < 0.01$$

example: $n = 9$

$$\binom{9}{7} p^7 (1-p)^2 + \binom{9}{8} p^8 (1-p) + \binom{9}{9} p^9$$

$$= 0.008281 < 0.01$$

example: $n = 10$

$$\binom{10}{7} p^7 (1-p)^3 \approx 0.016 > 0.01.$$

It's important to be able to read the English and map it into equivalent math.

Error correcting codes (example 2.40)

You send bits across a channel, but each bit arrives in error with probability of bit error $p = 10^{-3}$.

To increase the chance of correct delivery, you send the desired bit 3 times.

The decoder makes a majority decision (i.e., if 2 or 3 bits say 0, decide 0).

What's the probability of an incorrect decision?

Answer 3 independent Bernoulli trials,

event $A = \{$bit is received in error$\}$

Binomial $n = 3$, $p = P(A) = 10^{-3}$

$$P(\text{wrong decision}) = P(2 \text{ or more bit errors})$$

$$= P(2 \text{ bit errors}) + P(3 \text{ bit errors})$$

$$= \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3$$

$$\approx 3 * 10^{-6}$$

Error correcting codes are a major component of almost all communication systems. They can be made much more sophisticated than this simple "send in triplicate" strategy

Mean, variance, of binomial RV

Approach 1: A binomial RV can be considered to be the sum of n independent Bernoulli RVs.

if $X \sim \text{binomial}(n, p)$,

and $Y_i \sim \text{Bernoulli}(p)$ for $i = 1, \dots, n$,

then $X = Y_1 + Y_2 + \cdots + Y_n$

So it's easy to compute

$$
\begin{aligned}
E(X) &= E(Y_1 + Y_2 + \cdots + Y_n) \\
&= E(Y_1) + E(Y_2) + \cdots + E(Y_n) \\
&= p + p + \cdots + p \\
&= np
\end{aligned}
$$

Approach 2 (next page) uses a series summation

Approach 2 to compute $E(X)$ for binomial RV

$$E(X) = \sum_{k=0}^{n} k\, p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

expand

$$= \sum_{\underset{\text{can replace with 1}}{k=0}}^{n} k \underbrace{\frac{n!}{k!\,(n-k)!}}_{\text{reorganize}} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n\,(n-1)!}{(k-1)!\,(n-k)!} (p)\, p^{k-1} (1-p)^{n-k}$$

change of variables – substitute $j = k-1$ so $k = j+1$

$$= np \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!\,(n-1-j)!} p^j (1-p)^{n-1-j}}_{\text{this is the sum of pmf terms for a binomial RV w/ } (n-1,\, p)}$$

so

$$E(X) = np$$

Variance of a Binomial (n, p) RV

$$E(X^2) = \sum_{n=0}^{K} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{n=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

Let $j = k-1$ and $k = j+1$

$$= np \sum_{j=0}^{n-1} (j+1) \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j}$$

Break into 2 sums:

$$= np \sum_{j=0}^{n-1} j \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j}$$

$$+ np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j}$$

First term is np times $E(Y)$ where Y is binomial with $((n-1), p)$

Second term is np times the sum of a PMF.

$$= np\big((n-1)p + 1\big) = n^2p^2 - np^2 + np$$

Finally: $\mathrm{Var}(X) = E(X^2) - E(X)^2$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$\mathrm{Var}(X) = np(1-p)$$

n times the variance of a Bernoulli (p)