Topic 2.2b

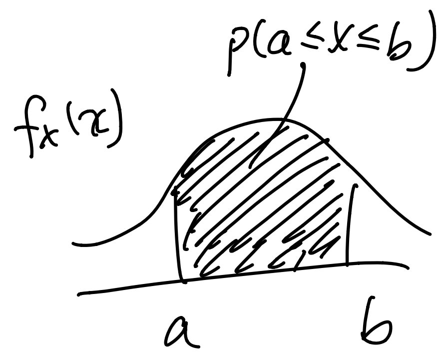
probability density function (chapter 4.2)

(except not Ch 4.2.2)

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Book claims $f_X(x)$ is more useful than $F_X(x)$.
I disagree. Both play a useful role.
Both provide the same information for a continuous R.V.

Properties of pdf $f_X(x)$



① $f_X(x) \geq 0$

② $P(a \le X \le b) = \int_a^b f_X(x)\,dx$

③ $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ ← by Fundamental thm of calculus

④ $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (by letting x go to ∞ in ③)

(warning - for ③ be careful of limit of integration and variable inside integral)

Interpretation of "density" in "pdf"

$$f_X(x) = \frac{d\,F_X(x)}{dx}$$

"density": what is $P(x < X \le x+h)$?

(the probability of a narrow interval about x of length h)

$$P(x < X \le x+h) = F_X(x+h) - F_X(x)$$

$$= \frac{F_X(x+h) - F_X(x)}{h} \cdot h$$

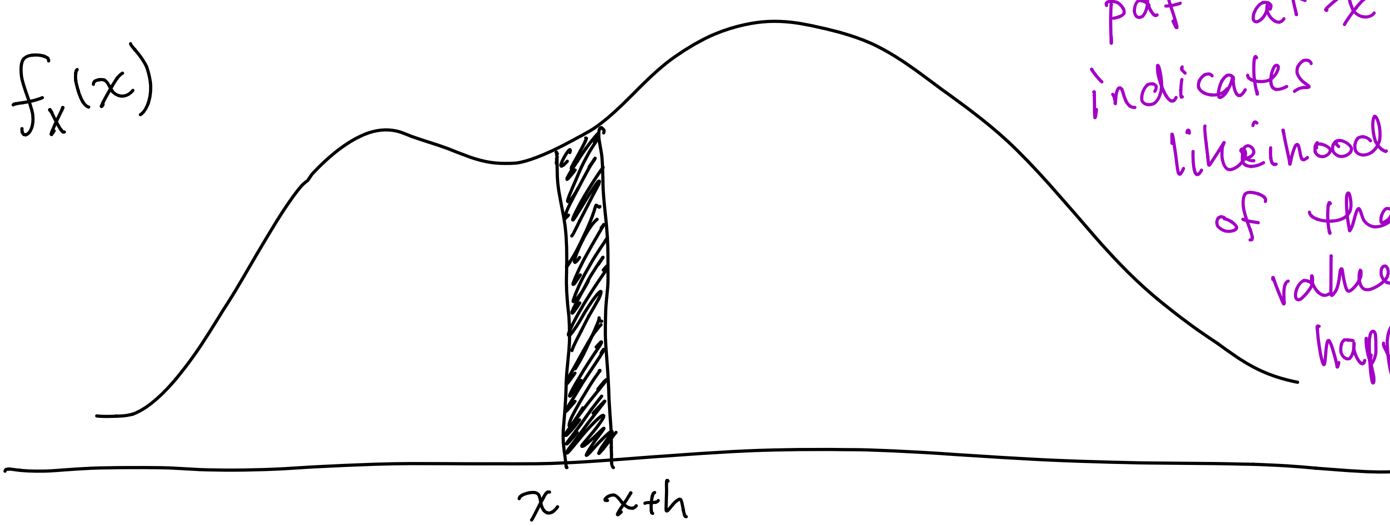
if $F_X(x)$ has a derivative,

then as $h \to 0$, this probability $\approx f_X(x) \cdot h$

so $f_X(x)$ is NOT a probability,

but $f_X(x) \cdot h$ is.

Intuition: height of pdf at x indicates likeihood of that value happening



Computing probabilities of events from a PDF or CDF.

Example: exponential R.V. (a good model for transmission time, waiting time, service time)

Suppose X is the transmission time of messages in a communication system, and

$$P(X > x) = e^{-\lambda x} \qquad \text{for } x \geq 0.$$

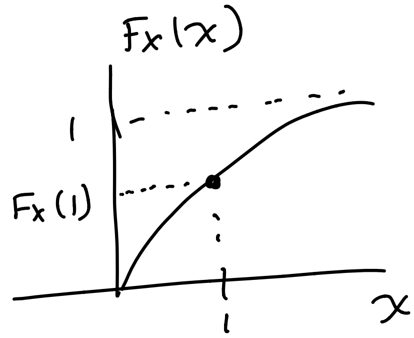
What are the CDF and PDF of X?

What is the probability that $X \leq 1$?

Answers:

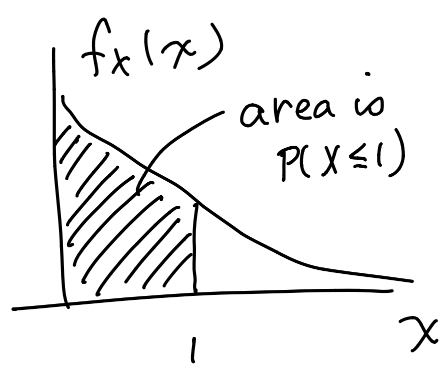
CDF $F_X(x) = P(X \leq x) = 1 - P(X > x)$

$$= \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



PDF $f_X(x) = \dfrac{d}{dx} F_X(x)$

$$= \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



method 1: $P(X \leq 1) = F_X(1) = \boxed{1 - e^{-\lambda}}$

method 2: $P(X \leq 1) = \int_{-\infty}^{1} f_X(x)\,dx = \int_0^1 \lambda e^{-\lambda x}\,dx$

$$= -e^{-\lambda x}\Big|_0^1 = -(e^{-\lambda} - 1) = \boxed{1 - e^{-\lambda}}$$

The properties of a pdf allow us to construct a useful probability model with almost any shape.

Ex: take a piecewise continuous $g(x) \geq 0 \ \forall x$ that satisfies $\int_{-\infty}^{\infty} g(x)\, dx = d$

for some $d < \infty$.

Then the function $f_X(x) = \frac{g(x)}{d}$ is a pdf

Example: $g(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & \text{else} \end{cases}$

Note that $\int_{-\infty}^{\infty} g(x)\, dx = \int_0^{1/2} 1 \cdot dx = x \Big|_0^{1/2} = \frac{1}{2} = c$

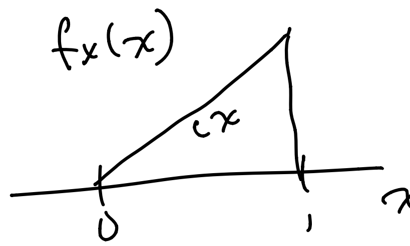
so $f_X(x) = 2g(x) = \begin{cases} 2 & 0 \leq x \leq 1/2 \\ 0 & \text{else} \end{cases}$

is a pdf.

Note that for this case $f_X(x) > 1$! $\boxed{\text{This is OK!}}$

(recall that $f_X(x)$ is not a probability)

Example: $f_X(x) = \begin{cases} cx & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$

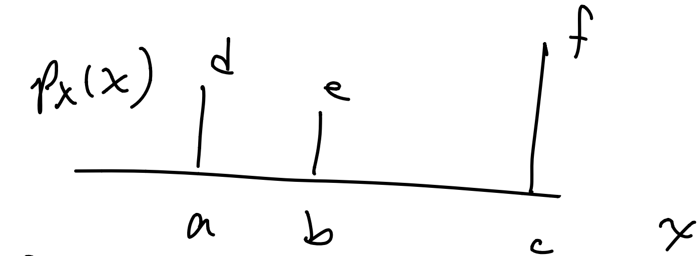


$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 cx\, dx = \frac{cx^2}{2}\Big|_0^1$

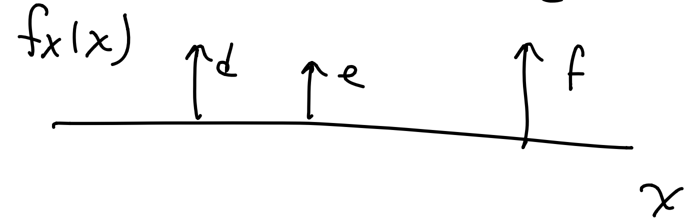
$= c/2 = 1$ $\Rightarrow$ to make $f_X(x)$ a valid pdf, set $\boxed{c = 2}$

PDF of discrete RVs.

PMF of X: $p_X(x)$



PDF of X: $f_X(x)$



Area of each delta function in $f_X(x)$ corresponds to the probability that $X=x$.

- A delta function in any PDF corresponds to a jump in the CDF.

Recall:
$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$\frac{d}{dx} u(x) = \delta(x)$$

$$u(x) = \int_{-\infty}^{x} \delta(t)\,dt$$

Sifting property
$$\int_{-\infty}^{\infty} \delta(x-a)\,g(x)\,dx = g(a)$$

Quick review: computing probabilities

pdf: $P(X \in A) = \int_{x \in A} f_X(x)\,dx$

cdf $P(X \le a) = F_X(a)$

$P(a < X \le b) = F_X(b) - F_X(a)$

$P(a \le X \le b) = \int_a^b f_X(x)\,dx$

$$P(a \le X \le b) = P(a < X \le b) + \underbrace{P(X = a)}_{\text{this is zero if } X \text{ is a continuous RV or if } F_X(x) \text{ is continuous at } x=a.}$$

this is zero if X is a continuous RV <u>or</u> if $F_X(x)$ is continuous at $x = a$.