Random Variables Chapter 3.1

Probability mass function (pmf) Ch 3.2

Cumulative distribution function (cdf) Ch 4.1

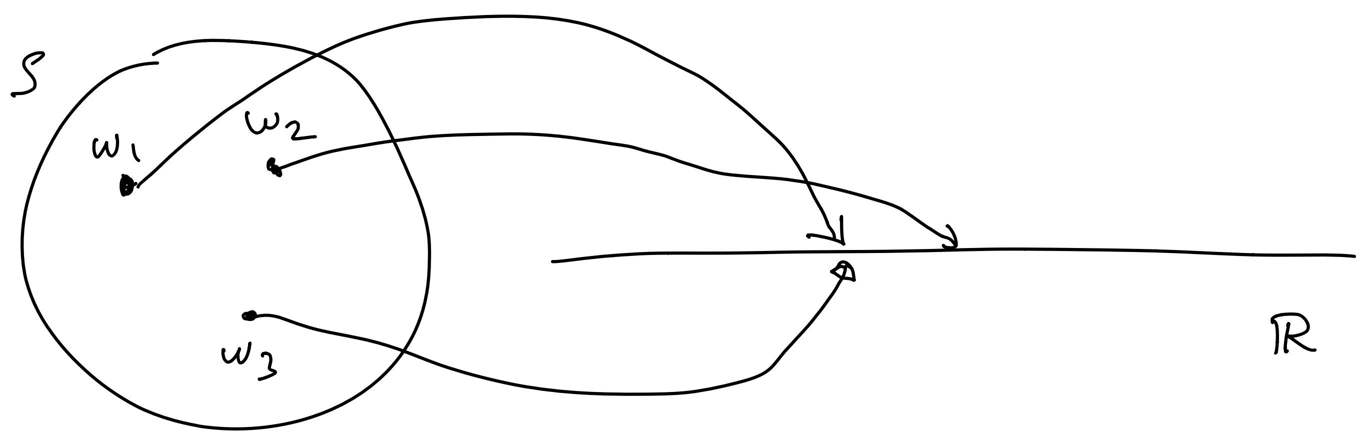
Probability density function (pdf) Ch 4.2

(except 4.2.2)

(Topic 2.1)

The outcome of a random experiment is often a number, so let's treat it like one.

- measurement on a tachometer
- stock price
- GPA
- power demand in a given city at a particular time of day in summer



$X(\omega)$: map the outcome from the sample space to a number on the real number line

a random variable (RV)

is a function : $X : S \to \mathbb{R}$

mapping $X(\omega) = x$ $\quad x \in \mathbb{R}$

X or $X(\omega)$ is a random variable

NOT a value

x is a value. It's not random at all.
It's just a number

ω is the outcome of the random experiment

$X(\omega)$ is a deterministic mapping (function)
– nothing random at all

Some Events of Interest using RVs

$$\{X(\omega) = x\}, \quad \{a \le X(\omega) < b\}$$

We still have our previous function

$$P : A \to [0,1] \qquad \text{(probability mapping)}$$

$$P(\{X(\omega) = x\}) = P(X = x)$$

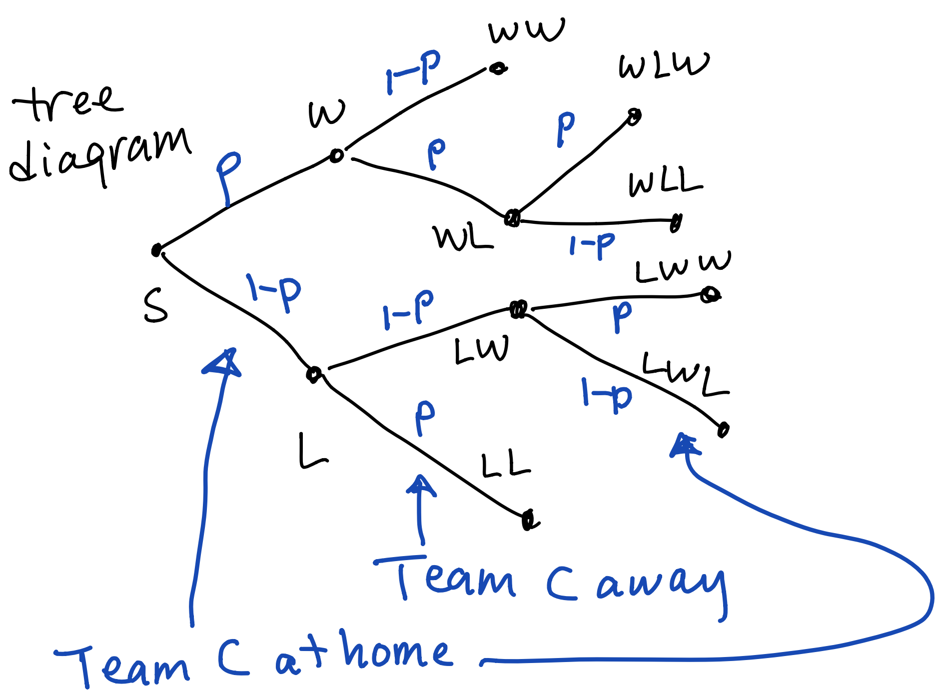
$$P(\{a \le X(\omega) \le b\}) = P(a \le X < b)$$

Probability that the random variable X takes on the value x.

Example : 3 game series

In a typical 3-game series, two teams play until one team wins 2 games. This either takes 2 or 3 games, depending on the outcome. Since there is often a "home game advantage", typically the venue is alternated to even the chances. Does this help?

Consider the case where team C plays first at home, then plays away and then, if necessary, plays at home. Let p = probability the home team wins.



Denote W to be a win by team C and L to be a loss by team C

Sample space $S = \{WW, WLW, WLL, LWW, LWL, LL\}$

2 RVs from this one experiment

let $X(\omega)$ = # games won by team C

Let $Y(\omega)$ = # games played to complete series

ω, outcomes	$P(\{\omega\})$ P(outcome)	$X(\omega)$	$Y(\omega)$
WW	$p(1-p)$	2	2
WLW	p^3	2	3
WLL	$p\cdot p\cdot(1-p)$	1	3
LWW	$(1-p)(1-p)p$	2	3
LWL	$(1-p)(1-p)(1-p)$	1	3
LL	$(1-p)p$	0	2

Can condense the table for both $X(\omega)$ and $Y(\omega)$
To do this, collect all possible outcomes $\{\omega : X = x\}$
and add their probabilities to get $p_X(x)$
(Remember, the outcomes form disjoint sets)

$X(\omega)$	$P(X(\omega) = x)$
0	$p(1-p)$
1	$p^2(1-p) + (1-p)^3$
2	$p(1-p) + p^3 + (1-p)^2 p$

A table indicating the probability mass function (pmf) for X

$Y(\omega)$	$P(Y(\omega) = y)$
2	$2p(1-p)$
3	$1-2p(1-p)$

The pmf of Y

Discrete or continuous?

If discrete, the range of X contains a countable (possibly infinite) number of elements $\{x_1, x_2, \ldots\}$

Discrete RVs have a probability mass function (pmf) $p_X(x)$

Continuous RVs can take on any real number y on an interval, ex: $a \le y \le b$

All RVs have a probability density function (pdf) $f_X(x)$ - see Chapter 4.2

All RVs have a cumulative distribution function (cdf) $F_X(x)$ - see Chapter 4.1

We'll start with discrete RVs

Probability mass function (pmf)

$p_X(x)$ = the probability of event $\{X(\omega) = x\}$

- lower case p, subscript capital X to denote which RV this is a pmf of, argument lower case x for the value
- pmf is only defined for discrete RVs.

$$p_X(x) = P(X = x) = P(\{\omega \in S : X(\omega) = x\})$$

for $x \in \mathbb{R}$

Calculating a pmf:

- For each value x that X can take, collect all possible outcomes $\{\omega : X(\omega) = x\}$ and add their probabilities to get $p_X(x)$

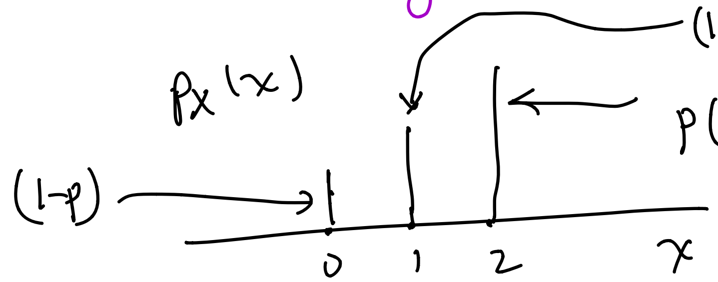
Can denote a pmf by

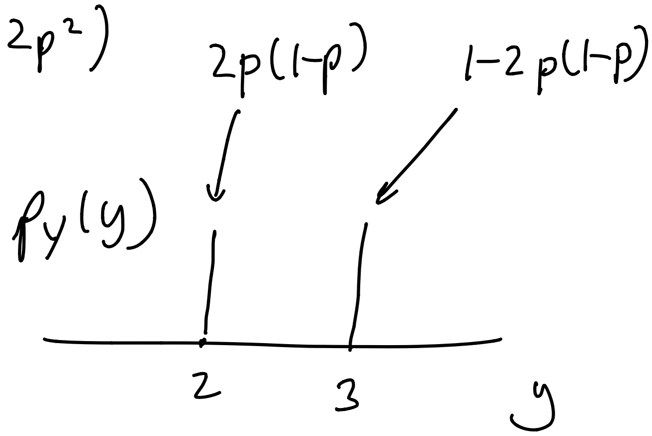
a) a table

b) a graph

c) a formula

From the 3-game series example





and using formulas

$$p_X(x) = \begin{cases} p(1-p) & \text{if } x=0 \\ (1-p)(p^2+(1-p)^2) & \text{if } x=1 \\ p(2-3p+2p^2) & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} 2p(1-p) & \text{if } y=2 \\ 1-2p(1-p) & \text{if } y=3 \\ 0 & \text{otherwise} \end{cases}$$

Always include the "otherwise" or "else" row!! The book doesn't always, but it's a useful reminder, and technically incomplete without it.

I WILL mark off if you omit it, and it's very easy to make subsequent mistakes if you forget it!

Another example

Roll a pair of fair dice and observe <u>maximum</u>

$S = \{ (i,j) : 1 \le i \le 6 \text{ and } 1 \le j \le 6 \} \quad (i,j \in \mathbb{Z})$

$Y(\omega) = Y((i,j)) = \max\{i,j\}$

What's the pmf of Y? $\qquad S_Y = \{1,2,3,4,5,6\}$

Enumerate sets:

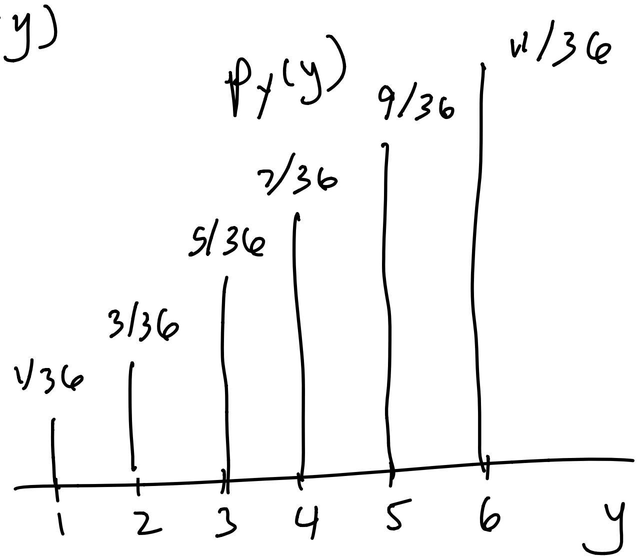
Set $\{Y=1\} = \{(1,1)\}$

Set $\{Y=2\} = \{(1,2),(2,1),(2,2)\}$

etc.

y	$p_Y(y) = P(Y=y)$
1	1/36
2	3/36
3	5/36
4	7/36
5	9/36
6	11/36
else	0

list sums to 1



Basic properties of a pmf $p_X(x)$

① $p_X(x) \geq 0$ for all x (recall: $p_X(x)$ is a probability)

② $\sum_{x \in S_X} p_X(x) = 1$ (because all the sets $\{X=x\}$ together form a partition of S)

③ $P(X \text{ is in event } B)$

$$= \sum_{x \in B} p_X(x)$$

(because of Axiom III)

Summary and preview

- We define a RV as a function of outcomes
- $P(\cdot)$ is still a mapping from events to $[0,1]$
- There are 3 ways we can get a RV
  - The RV is the observation (ex: photon count)
  - The RV is a function of the outcome (# working circuits)
  - The RV is a function of another RV (power from voltage, or revenue from working circuits)

- We can condition a RV on an event or a RV
- We will explore independence of an RV from an event or another RV