Sequential Experiments (Chapter 2.6)

(Sections 2.6.1 and 2.6.5 only)

Many experiments are actually a sequence of subexperiments.
These may be dependent or independent subexperiments.

Dependent : the procedure of a subexperiment depends on the previous outcome(s).

Example: roll a die, then flip that many coins and count the number of heads

Example: flip a coin until you get one head.

Independent : The procedure of each subexperiment does not depend on any previous outcome.

Example: flip a coin 100 times, count # heads

Example: roll 5 dice and sum the dots shown

A tree diagram is an effective solution method

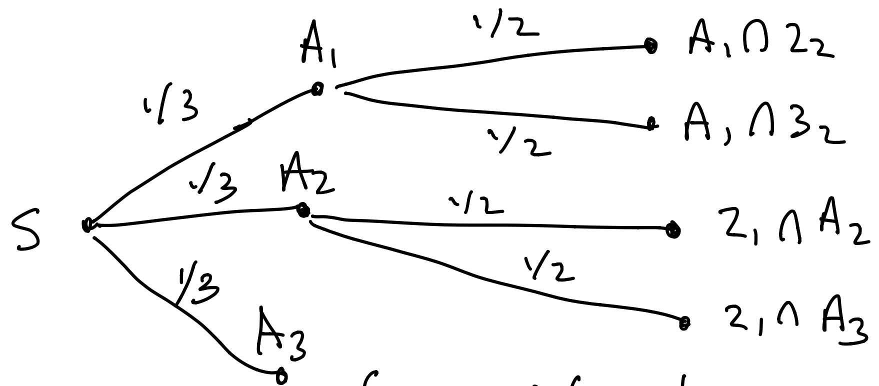
Example of sequential dependent experiments

The "game of three" uses a deck of 3 cards: Ace, 2, 3. The Ace is worth 1 point. Draw cards (without replacement) until your total is 3 or more. You win if you get 3 exactly. What is the probability you win?

Answer: $W = \{\text{win}\}$, want $P(W)$.

Let $C_i = \{\text{card } C \text{ is } i^{th} \text{ card drawn}\}$

Example: 3_2 is the event the 3 is drawn 2nd.



(no more branches experiment stops)

$$W = (A_1 \cap 2_2) \cup (2_1 \cap A_2) \cup 3_1$$

$$P(W) = P(A_1 \cap 2_2) + P(2_1 \cap A_2) + P(3_1)$$

$$= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

Example of sequential dependent experiments

Engineers coordinating 2 traffic lights to make the probability some one gets the same color light at the second light that they got at the first light is 0.8

$R_1 = \{\text{first light red}\}$ $\quad G_1 = R_1^c$ $\quad$ (no yellow lights)

$R_2 = \{\text{second light red}\}$ $\quad G_2 = R_2^c$

Given: $P(R_2 | R_1) = 0.8$ $\quad P(G_2 | R_1) = 0.2$

$P(G_2 | G_1) = 0.8$ $\quad P(R_2 | G_1) = 0.2$

<u>Questions</u>: a) Assume 1^{st} light equally likely to be red or green. What's the probability the second light is green?

Answer

Given $P(R_1) = P(G_1) = 1/2$

want $P(G_2) = P(G_2 | R_1) P(R_1) + P(G_2 | G_1) P(G_1)$

by theorem of total probability, since R_1 and G_1 form a partition

$P(G_2) = (0.2)(0.5) + (0.8)(0.5) = 0.5$

b) What's the probability a driver has to wait for at least one light?

Answer

$= 1 - P(\text{no waiting}) = 1 - P(G_1 \cap G_2)$

$= 1 - P(G_2 \mid G_1)\,P(G_1) = 1 - (0.8)(0.5) = 0.6$

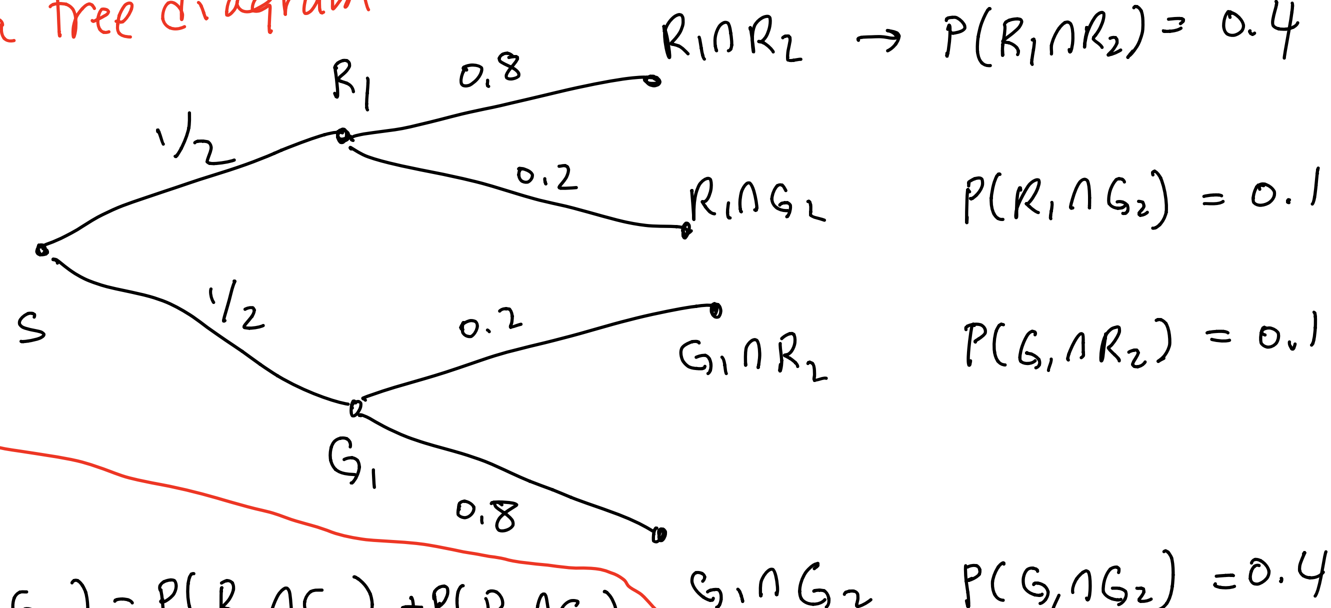
c) Given that the driver has to wait at the second light, what's the probability the first light was green?

Answer

want $\quad P(G_1 \mid R_2) = \dfrac{P(R_2 \mid G_1)\,P(G_1)}{P(R_2)}$

$$= \frac{(0.2)(0.5)}{1 - P(G_2)} = 0.2$$

with a tree diagram



a) $P(G_2) = P(R_1 \cap G_2) + P(R_2 \cap G_2)$

$= 0.4 + 0.1 = 1/2$

etc.

Example of sequential dependent experiments

Monty Hall problem. "Let's make a deal"

There are 3 doors. Behind one of the doors is a car. Behind each of the other two is a goat.

Step 1: pick a door

Step 2: Host opens one of the other doors and shows you a goat.

Now 2 doors are closed and one is open. Behind one closed door is a goat, behind the other is a car.

Step 3: Host asks if you want to switch doors, or stay with the door you first picked.

Question: Should you switch? Does it matter?

Assumptions: a) the car is not moved
b) the host knows where the car is and will never open that door in step 2.

Incorrect intuition: There are 2 doors left and one car, so each door has a $1/2$ chance of hiding the car. So there's not need to switch.
This is incorrect!

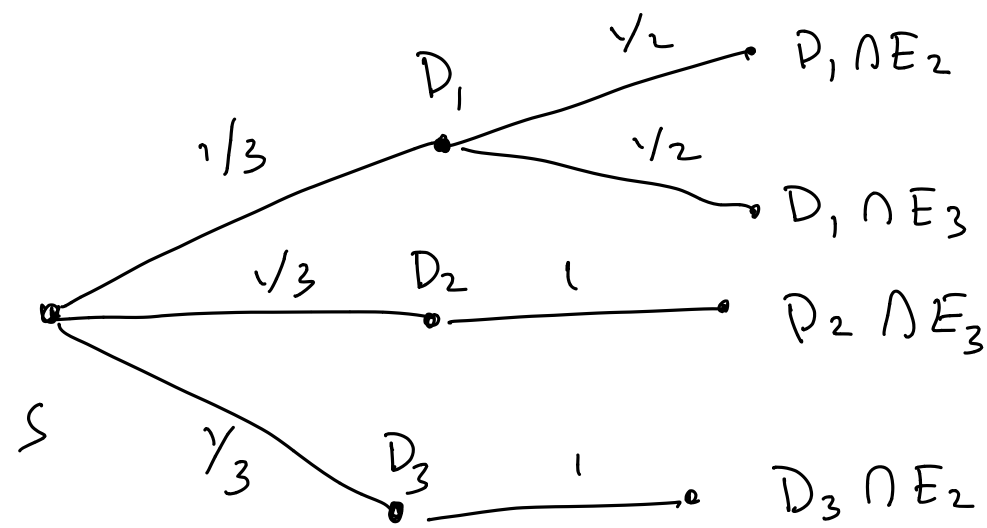
Analyze Monty Hall as a sequential experiment.

Let $D_i = \{$ car is behind Door $i \}$ $\quad i = 1, 2, 3$

$E_j = \{$ host Exposes door $j \}$ $\quad j = 1, 2, 3$

we know $D_k \cap E_k = \phi$ for any k.

without loss of generality, we can assume you always choose Door 1. so $j = 2, 3$ and never $j = 1$.



$P(D_1 \cap E_2) = 1/6$

$P(D_1 \cap E_3) = 1/6$

$P(D_2 \cap E_3) = 1$

$P(D_3 \cap E_2) = 1$

4 possible outcomes, but not all are equally likely.

$$P(D_1) = P(D_1 \cap E_2) + P(D_1 \cap E_3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P(D_1^c) = P(D_2 \cap E_3) + P(D_3 \cap E_2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$\Rightarrow$ **SWITCH** for a higher chance of winning car