Conditional Probability (Chapter 2.4)

Two useful applications of conditional probability

1. Build complex probability models from simpler pieces

3. Learn from the result of the experiment

The math: Three basic equations

Definition $$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

Theorem of Total probability $$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$ if the B_i's form a partition

Bayes Rule $$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \quad \text{if } P(A) > 0$$

You are expected to memorize these and know how to apply them

An example that leads into conditional probability

Mixed electronic components in a store.
Cannot buy separately, must buy in bulk.

29% bad parts
3% bad resistors
12% good resistors
5% bad capacitors
32% diodes

① Define observations: each part has a type and a good/bad.

② Define partitions: ⓐ R, C, D where

$R = \{\text{resistors}\}$, $C = \{\text{capacitors}\}$, $D = \{\text{diodes}\}$

ⓑ B, B^c where $B = \{\text{bad}\}$, $B^c = \{\text{good}\}$.

③ build probability table and complete it

	R	C	D	
B	0.03	0.05		0.29
B^c	0.12			
			0.32	

	R	C	D	
B	0.03	0.05	0.21	0.29
B^c	0.12	0.48	0.11	0.71
	0.15	0.53	0.32	

Answer questions:

Q1) What's the probability a random component is a resistor?

$$P(R) = P(R \cap B) + P(R \cap B^c)$$
$$= 0.12 + 0.03 \quad = 0.15$$

Q2) Suppose you have no use for either bad parts <u>or</u> resistors.
What's the probability a random part is useful to you?

Event "useful" $= (B \cup R)^c$

(because event "not useful" is $B \cup R$)

$$P((B \cup R)^c) = 1 - P(B \cup R)$$
$$= 1 - [P(B) + P(R) - P(B \cap R)]$$
$$= 1 - [0.29 + 0.15 - 0.03] = 1 - .41 = 0.59$$

Q3) What's the probability of a bad diode?

Want $P(B \cap D)$.

From Thm total prob, $P(B) = P(B \cap R) + P(B \cap C) + P(B \cap D)$

or $0.29 = 0.03 + 0.05 + P(B \cap D) \implies P(B \cup D) = 0.21$

These questions have all been review questions

Q4) Suppose you pull out a resistor. what's the probability it's bad?

The 1st experiment (for Q1-Q3) implicitly looked at two things simultaneously:

part type and part good/bad

With this question, we already know the part is a resistor — so it's a <u>different experiment</u>.

We can use our probability model for the 1st experiment to create one for the 2nd experiment

> Restrict the sample space of the 1st experiment to get a new sample space where we <u>know</u> that we have a resistor.

Now only 2 outcomes can happen:

A good part (that's a resistor) and

a bad part (that's a resistor)

In this new experiment, the sum of the probabilities of these outcomes must be 1.

1st experiment

	R	C	D
B	0.03		
G	0.12		
	0.15		

2nd experiment

	R
B	?
G	?
	1

We denote the probability a part is bad given that it is a resistor as

$P(B|R)$.

And we compute it by rescaling the probabilities:

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.03}{0.15} = \frac{1}{5}$$

$$P(B^c|R) = \frac{P(B^c \cap R)}{P(R)} = \frac{0.12}{0.15} = \frac{4}{5}$$

Two steps: 1 narrow the sample space
2 rescale so all probabilities in new sample space sum to 1

Q5) Suppose the components are sorted into Good and Bad, and you pull a component at random from the Good pile.

What's the probability a random Good component is a capacitor?

$$P(C|G) = \frac{P(C \cap G)}{P(G)} = \frac{0.48}{0.71}$$

This leads us to the formal definition of
conditional probability

Suppose 2 events, A and B, are defined on the same sample space.

Then $P(A|B)$ is the conditional probability of event A conditioned on (or given) event B,

and $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) \neq 0$

As stated above, there are two useful things we can accomplish with conditional probability:

1. build more complicated (and accurate) models by combining simpler models (using the Theorem of Total probability)
2. learn by observing the outcome of an experiment (using Bayes Rule)

English and Conditional probability

Fall 2016
(September 6, 2016)

Conditional, or joint probability??

- Suppose you pull out a part, what's the probability it's a bad resistor?
- Suppose you pull out a part, what's the probability it's bad and a resistor?
- Suppose you pull out a resistor, what's the probability that it's bad?
- What's the probability that the part is bad, given that it's a resistor?
- What's the probability you pull out a bad resistor?
- What's the probability a resistor is bad?
- What's the probability a bad part is a resistor?
- What's the probability a part is a bad resistor?

English and Conditional probability

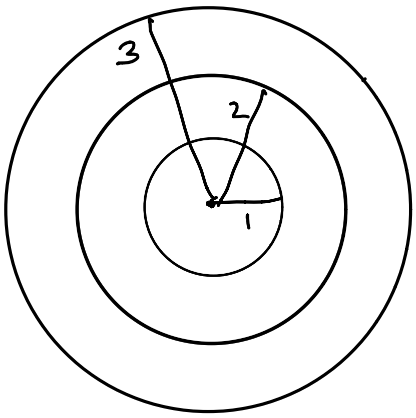
Fall 2016
(September 6, 2016)

Conditional, or joint probability??

- $B \cap R$ Suppose you pull out a part, what's the probability it's a bad resistor? joint
- $B \cap R$ Suppose you pull out a part, what's the probability it's bad and a resistor? joint
- $B|R$ Suppose you pull out a resistor, what's the probability that it's bad? conditional
- $B|R$ What's the probability that the part is bad, given that it's a resistor? conditional
- $B \cap R$ What's the probability you pull out a bad resistor? joint
- $B|R$ What's the probability a resistor is bad? conditional
- $R|B$ What's the probability a bad part is a resistor? conditional
- $B \cap R$ What's the probability a part is a bad resistor? joint

An example of conditional probability with a continuous sample space

Throw a dart at a dart board.
Assume the probability it misses the board is zero.
Assume equally likely to hit anywhere on the board.
Suppose dart board has radius 3.



$S = \left\{ \begin{array}{l} \text{location dart} \\ \text{hits on dart board} \end{array} \right\}$

$B = \left\{ \begin{array}{l} \text{dart hits inside} \\ \text{circle w/ radius 2} \end{array} \right\}$

$A = \left\{ \begin{array}{l} \text{dart hits inside} \\ \text{circle w/ radius 1} \end{array} \right\}$

$$P(A) = \frac{\text{area}(A)}{\text{area}(S)} = \frac{\pi}{9\pi} = \frac{1}{9}$$

because all points in S are equally likely

$$P(B) = \frac{\text{area}(B)}{\text{area}(S)} = \frac{4\pi}{9\pi} = \frac{4}{9}$$

$$P(A \cap B) = P(A) = 1/9$$

$$P(A|B) = \frac{\text{area}(A)}{\text{area}(B)} = \frac{\pi}{4\pi} = 1/4$$

using the intuition that we can ignore the parts of the board that are not in B

or using definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/9}{4/9} = 1/4$$

Conditional probability satisfies the 3 axioms.

	probability	conditional probability
I	$P(S) = 1$	$P(B \mid B) = 1$
II	$P(A) \geq 0$	$P(A \mid B) \geq 0$
III	if $A_1 \cap A_2 = \phi$ and $A = A_1 \cup A_2$, then $P(A) = P(A_1) + P(A_2)$	if $A_1 \cap A_2 = \phi$, and $A = A_1 \cup A_2$ then $P(A \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$