Name: ______________________________ PU ID: ____________

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

Homework 8

Fall 2021
(Due Thursday October 28, 11:59pm)

Homework is due on **Thursday October 28 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

Statement:

Topics: Hypothesis testing (lecture notes Topic 2.7); Joint PDF and CDF and PMF, Marginal PDF and CDF and PMF (Ch 5.1-5.4); Independence (Ch 5.5)

Exercise 1. (THIS PROBLEM IS WORTH 2 POINTS)
A constant signal S may (or may not) be sent over a channel with additive noise, N. Let H_1 denote the event that the signal was sent, and H_0 denote the event that no signal is sent. The goal at the receiver is to decide which event occured, either H_1 or H_0. If the signal is sent (event H_1), the random variable X at the receiver is $X = S + N$, but if the signal is not sent (event H_0), then $X = N$.

Let the constant signal be $S = 1$, and let N be a random variable with a triangular PDF, between -1 and 1. Specifically,

$$f_N(y) = \begin{cases} y+1 & \text{for } -1 \le y < 0 \\ (1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding CDF is

$$F_N(y) = \begin{cases} 0 & \text{for } y < -1 \\ (1+2y+y^2)/2 & \text{for } -1 \le y < 0 \\ (1+2y-y^2)/2 & \text{for } 0 \le y < 1 \\ 1 & \text{for } y \ge 1 \end{cases}$$

(a) Sketch the conditional PDFs *and* CDFs: $f_X(x|H_0)$, $f_X(x|H_1)$, $F_X(x|H_0)$, and $F_X(x|H_1)$.

(more space)

(b) Because the PDFs of these two situations overlap, it is not possible to design an error-free detector; any actual system will always have a non-zero probability of error. In radar systems, two quantities are of interest. The first is the *probability of false alarm*, where the receiver detects a signal is present (i.e., decides H_1) when in fact no signal was present. The second is the *probability of detection*, where the receiver correctly detects a signal is present (i.e., decides H_1 when H_1 is true).

Suppose the detector is designed to decide H_1 when the received value of X is greater than a constant threshold t. Express the probabilities of false alarm and detection as a function of either the conditional PDF or conditional CDF of X given the associated event (i.e., H_1 or H_0). Your answers will depend on the threshold t.

(c) Consider the area where the PDFs overlap, namely $0 \leq x \leq 1$. And suppose that the prior probabilities $P(H_0) = 1/4$ and $P(H_1) = 3/4$. Find the threshold t which will minimize the *probability of error* (Hint: be sure to account for the two types of errors: (a) deciding H_0 when H_1 is true, and (b) deciding H_1 when H_0 is true.)

Exercise 2. (Not quite textbook, problems 5.1)
Let X be the maximum and Y be the minimum of the number of heads obtained when Carlos and Mihaela each flip a biased coin twice; P(heads)=3/4.

(a) Describe the sample space S of two sets of coin tosses, and the sample space $S_{X,Y}$ of these two random variables, showing how S is related to $S_{X,Y}$.

(b) Find the probabilities for all values of (X, Y).

(c) Find $P(X = Y)$.

Exercise 3. (FROM TEXTBOOK, PROBLEMS 5.17)
A point (X, Y) is selected at random inside a triangle defined by $\{(x, y) : 0 \leq y \leq x \leq 1\}$. Assume the point is equally likely to fall anywhere in the triangle.
It is possible to show that the joint CDF for these RVs is

$$F_{XY}(x, y) = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ 2xy - y^2 & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x \\ x^2 & 0 \leq x \leq 1 \text{ and } x \leq y \\ 2y - y^2 & 1 \leq x \text{ and } 0 \leq y \leq 1 \\ 1 & 1 \leq x \text{ and } 1 \leq y \end{cases}$$

(a) Find the marginal CDF of X and of Y.

(b) Use the joint CDF to find the probabilities of the following events:
$A = \{X \leq 2, Y \leq 3/4\}$, $B = \{1/4 < X \leq 3/4, 1/4 < Y \leq 3/4\}$

(c) Apply double integration to compute these same probabilities using the joint PDF.

Exercise 4. (From textbook, combining problems 5.26, 5.65, and 5.80)
Let X and Y have the joint PDF

$$f_{X,Y}(x, y) = k(x + y), \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1$$

(a) Find k.

(b) Find the marginal PDF of X and of Y.