Name: ______________________________ PU ID: ______________

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Homework 8

Fall 2021
(Due Thursday October 28, 11:59pm)

Homework is due on **Thursday October 28 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

Statement:

Topics: Hypothesis testing (lecture notes Topic 2.7); Joint PDF and CDF and PMF, Marginal PDF and CDF and PMF (Ch 5.1-5.4); Independence (Ch 5.5)

Exercise 1. (THIS PROBLEM IS WORTH 2 POINTS)
A constant signal S may (or may not) be sent over a channel with additive noise, N. Let H_1 denote the event that the signal was sent, and H_0 denote the event that no signal is sent. The goal at the receiver is to decide which event occured, either H_1 or H_0. If the signal is sent (event H_1), the random variable X at the receiver is $X = S + N$, but if the signal is not sent (event H_0), then $X = N$.

Let the constant signal be $S = 1$, and let N be a random variable with a triangular PDF, between -1 and 1. Specifically,

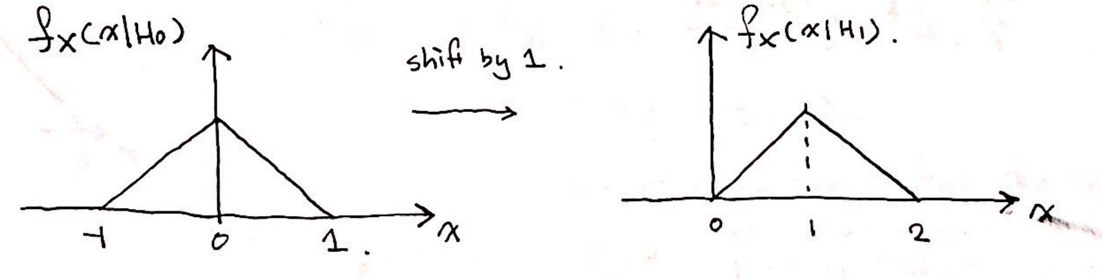
$$f_N(y) = \begin{cases} y+1 & \text{for } -1 \le y < 0 \\ (1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding CDF is

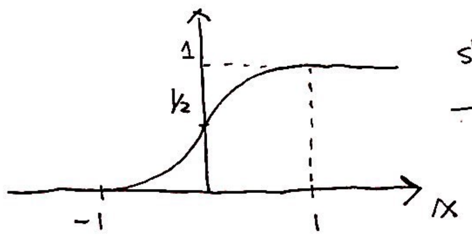
$$F_N(y) = \begin{cases} 0 & \text{for } y < -1 \\ (1+2y+y^2)/2 & \text{for } -1 \le y < 0 \\ (1+2y-y^2)/2 & \text{for } 0 \le y < 1 \\ 1 & \text{for } y \ge 1 \end{cases}$$

(a) Sketch the conditional PDFs *and* CDFs: $f_X(x|H_0)$, $f_X(x|H_1)$, $F_X(x|H_0)$, and $F_X(x|H_1)$.

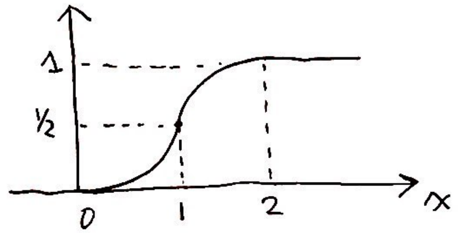
$H_0: X = N$ $\qquad$ $H_1: X = N + S$, where $S = 1$.



$F_X(x|H_0)$. (more space) $F_X(x|H_1)$.



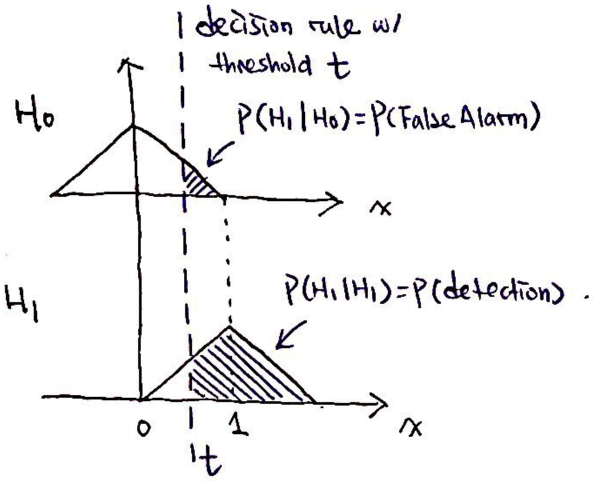
shift by 1.



(b) Because the PDFs of these two situations overlap, it is not possible to design an error-free detector; any actual system will always have a non-zero probability of error. In radar systems, two quantities are of interest. The first is the *probability of false alarm*, where the receiver detects a signal is present (i.e., decides H_1) when in fact no signal was present. The second is the *probability of detection*, where the receiver correctly detects a signal is present (i.e., decides H_1 when H_1 is true).

Suppose the detector is designed to decide H_1 when the received value of X is greater than a constant threshold t. Express the probabilities of false alarm and detection as a function of either the conditional PDF or conditional CDF of X given the associated event (i.e., H_1 or H_0). Your answers will depend on the threshold t.

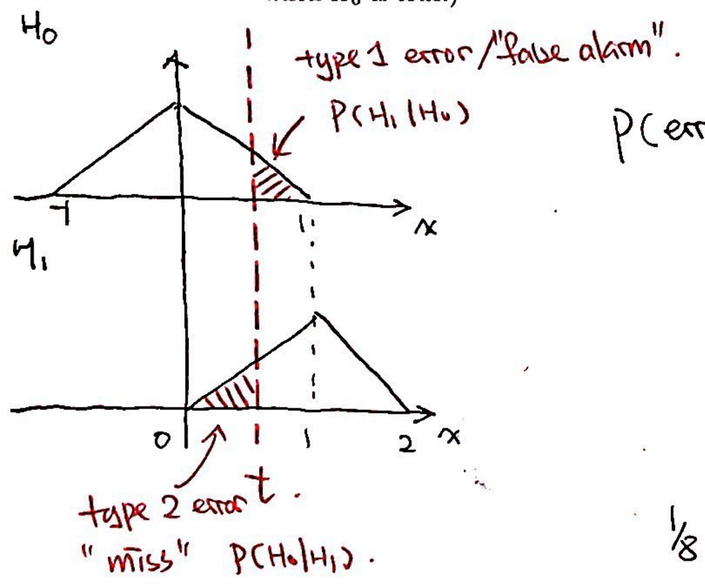
$-1 \le t \le 1$ ✓



$$P(\text{False Alarm}) = P(H_1|H_0) = \begin{cases} 1 - F_X(t|H_0) . \\ 0 , & t > 1 \\ 1 , & t < -1 . \end{cases}$$

$$P(\text{detection}) = P(H_1|H_1) = \begin{cases} 1 - F_X(t|H_1), & 0 \le t \le 2 . \\ 0 , & t > 2 \\ 1 , & t < 0 . \end{cases}$$

(c) Consider the area where the PDFs overlap, namely $0 \le x \le 1$. And suppose that the prior probabilities $P(H_0) = 1/4$ and $P(H_1) = 3/4$. Find the threshold t which will minimize the *probability of error* (Hint: be sure to account for the two types of errors: (a) deciding H_0 when H_1 is true, and (b) deciding H_1 when H_0 is true.)



For range $0 \le t \le 1$:

$$\begin{aligned} P(\text{error}) &= P(H_0) \cdot P(H_1|H_0) + P(H_1) \cdot P(H_0|H_1) . \\ &= \tfrac{1}{4} \cdot (1 - F_X(t|H_0)) + \tfrac{3}{4} \cdot F_X(t|H_1) . \\ &= \tfrac{1}{4} \cdot \left(1 - \frac{1+2t-t^2}{2}\right) + \tfrac{3}{4} \cdot \frac{t^2}{2} . \\ &= \tfrac{1}{8}(4t^2 - 2t + 1) . \end{aligned}$$

↓ take derivative & find the root.

$\frac{1}{8}(8t-2) = t - \frac{1}{4} = 0$: $\boxed{t = \tfrac{1}{4} \text{ gives us minimum error}}$

Exercise 2. (NOT QUITE TEXTBOOK, PROBLEMS 5.1)
Let X be the maximum and Y be the minimum of the number of heads obtained when Carlos and Mihaela each flip a biased coin twice; P(heads)=3/4.

(a) Describe the sample space S of two sets of coin tosses, and the sample space $S_{X,Y}$ of these two random variables, showing how S is related to $S_{X,Y}$.

(b) Find the probabilities for all values of (X, Y).

(c) Find $P(X = Y)$.

a.)

Carlos (columns), Mihaela (rows)

(X,Y).	HH	HT	TH	TT
HH	(2,2)	(2,1)	(2,1)	(2,0)
HT	(2,1)	(1,1)	(1,1)	(1,0)
TH	(2,1)	(1,1)	(1,1)	(1,0)
TT	(2,0)	(1,0)	(1,0)	(0,0)

S = { (HH,HH), (HH,HT), (HH,TH), (HH,TT),
(HT,HH), (HT,HT), (HT,TH), (HT,TT),
(TH,HH), (TH,HT), (TH,TH), (TH,TT),
(TT,HH), (TT,HT), (TT,TH), (TT,TT) }.

$S_{XY} = \{(2,2), (2,1), (2,0), (1,1), (1,0), (0,0)\}$.

b.)

$P_{XY}(2,2) = (3/4)^4 = 81/256$.

$P_{XY}(2,1) = (3/4 \cdot 1/4) \cdot (3/4 \cdot 3/4) \cdot 4 = 108/256$.

$P_{XY}(2,0) = (1/4 \cdot 1/4) \cdot (3/4 \cdot 3/4) \cdot 2 = 18/256$.

$P_{XY}(1,1) = (3/4 \cdot 1/4) \cdot (3/4 \cdot 1/4) \cdot 4 = 36/256$.

$P_{XY}(1,0) = (1/4 \cdot 1/4) \cdot (3/4 \cdot 1/4) \cdot 4 = 12/256$.

$P_{XY}(0,0) = (1/4 \cdot 1/4) \cdot (1/4 \cdot 1/4) = 1/256$.

c.) $P(X=Y) = P_{XY}(0,0) + P_{XY}(1,1) + P_{XY}(2,2)$

$= 118/256$.

Exercise 3. (FROM TEXTBOOK, PROBLEMS 5.17)
A point (X, Y) is selected at random inside a triangle defined by $\{(x, y) : 0 \le y \le x \le 1\}$. Assume the point is equally likely to fall anywhere in the triangle.
It is possible to show that the joint CDF for these RVs is

$$F_{XY}(x,y) = \begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ 2xy - y^2 & 0 \le x \le 1 \text{ and } 0 \le y \le x \\ x^2 & 0 \le x \le 1 \text{ and } x \le y \\ 2y - y^2 & 1 \le x \text{ and } 0 \le y \le 1 \\ 1 & 1 \le x \text{ and } 1 \le y \end{cases}$$

(a) Find the marginal CDF of X and of Y.

(b) Use the joint CDF to find the probabilities of the following events:
$A = \{X \le 2, Y \le 3/4\}$, $B = \{1/4 < X \le 3/4, 1/4 < Y \le 3/4\}$

(c) Apply double integration to compute these same probabilities using the joint PDF.

a).

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = \begin{cases} 0, & x \le 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

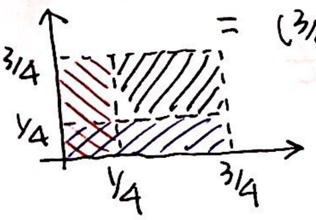
$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y) = \begin{cases} 0, & y \le 0 \\ 2y-y, & 0 \le y \le 1 \\ 1, & y \ge 1. \end{cases}$$

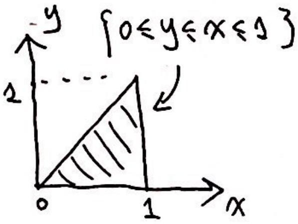
b).

$$P(A) = P(X \le 2, Y \le 3/4) = F_{XY}(2, 3/4) = 2 \cdot 3/4 - (3/4)^2 = \boxed{15/16.}$$

$$P(B) = F_{XY}(3/4, 3/4) - F_{XY}(3/4, 1/4) - F_{XY}(1/4, 3/4) + F_{XY}(1/4, 1/4).$$

$$= (3/4)^2 - (1/4)^2 - 2 \cdot 1/4 \cdot 3/4 + (1/4)^2 + (1/4)^2 = \boxed{1/4.}$$

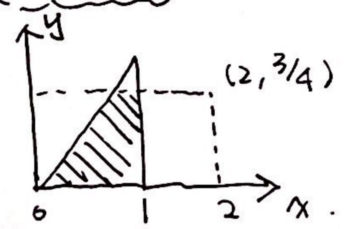




c).

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \begin{cases} 0, & 0 \le x \text{ or } y \le 0. \\ 2, & 0 \le x \le 1 \text{ and } 0 \le y \le x \\ 0, & 0 \le x \le 1 \text{ and } x \le y \\ 0, & 1 \le x \text{ and } 0 \le y \le 1 \\ 0, & 1 \le x \text{ and } 1 \le y. \end{cases} = 2, \text{ for } 0 \le y \le x \le 1.$$

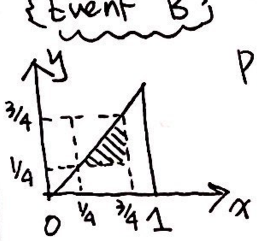
{Event A:}



$$P(A) = 1 - \int_{3/4}^{1} \int_{3/4}^{y} 2 \cdot dy dx$$

$$= 1 - 1/16 = \boxed{15/16}.$$

{Event B}



$$P(B) = \int_{1/4}^{3/4} \int_{1/4}^{x} 2 \, dy dx$$

$$= \boxed{1/4}.$$

Exercise 4. (FROM TEXTBOOK, COMBINING PROBLEMS 5.26, 5.65, AND 5.80)
Let X and Y have the joint PDF

$$f_{X,Y}(x, y) = k(x + y), \text{ for } 0 \leq x \leq 1, 0 \leq y \leq 1$$

(a) Find k.

(b) Find the marginal PDF of X and of Y.

a). for $f_{XY}(x,y)$ to be a valid pdf, $\iint_{\mathbb{R}^2} f_{XY}(x,y)\,dxdy = 1$.

$\therefore \int_0^1\int_0^1 k(x+y) \overset{dxdy}{\vee} = k\int_0^1 \frac{1}{2} + y\, dy = k \cdot 1 = 1$

$\therefore k = 1$.

b). $f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = \int_0^1 x+y\,dy$

$= \frac{1}{2} + x \,;\quad 0 \leq x \leq 1$

$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx = \int_0^1 x+y\,dx$

$= \frac{1}{2} + y \,.\quad 0 \leq y \leq 1$.