

Name: \_\_\_\_\_

PU ID: \_\_\_\_\_

ECE 302: Probabilistic Methods in Electrical and Computer Engineering  
Fall 2021

Instructor: Prof. A. R. Reibman



## Homework 7

Fall 2021

(Due Thursday October 21, 11:59pm)

Homework is due on **Thursday October 21 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

### Statement:

Topics: Common PDFs (Ch 4.4); Functions of a Random Variable (Ch 4.5)

Exercise 1. (FROM TEXTBOOK, PROBLEM 4.63 (A-C))

Let  $X$  be a Gaussian random variable with mean 5 and variance 16.

$$\sigma^2 = 16 \rightarrow \sigma = 4$$

(a) Find  $P(X > 4)$ ,  $P(X \geq 7)$ ,  $P(2 < X < 7)$ ,  $P(6 \leq X \leq 8)$ .

(b) If  $P(X < a) = 0.8869$ , what is the value of  $a$ ? standard Gaussian

(c) If  $P(X > b) = 0.11131$ , what is the value of  $b$ ?

$$a.) P(X > 4) = 1 - P(X \leq 4) = 1 - P\left(\frac{X-5}{4} \leq \frac{4-5}{4}\right) = 1 - \Phi(-0.25) \approx 0.5987$$

$$P(X \geq 7) = 1 - P(X < 7) = 1 - P\left(\frac{X-5}{4} < \frac{7-5}{4}\right) = 1 - \Phi(0.5) \approx 0.3085$$

$$P(2 < X < 7) = P(X < 7) - P(X < 2) = \Phi(0.5) - \Phi(-0.75) \approx 0.4649$$

$$P(6 \leq X \leq 8) = P(X \leq 8) - P(X \leq 6) = \Phi(0.75) - \Phi(0.25) \approx 0.1747.$$

$$b.) P(X < a) = P\left(\frac{X-5}{4} < \frac{a-5}{4}\right) = \Phi\left(\frac{a-5}{4}\right) = 0.8869$$

$$\therefore \frac{a-5}{4} = 1.21 \rightarrow a = 9.84$$

$$c.) P(X > b) = 1 - P(X \leq b) = 1 - P\left(\frac{X-5}{4} \leq \frac{b-5}{4}\right) = 0.11131$$

$$\therefore 1 - \Phi\left(\frac{b-5}{4}\right) = 0.11131, \quad \Phi\left(\frac{b-5}{4}\right) = 0.88869$$

$$\frac{b-5}{4} \approx 1.22, \quad b = 9.88$$

**Exercise 2.**

Let  $X$  be a Gaussian variable with  $E(X) = 0$  and  $P(|X| \leq 10) = 0.1$ . What is the standard deviation of  $X$ ? You may want to use the facts that  $\Phi(-1.28) = 0.1$ ,  $\Phi(-1.64) = 0.05$ , and  $\Phi(-1.96) = 0.025$ .

$$\begin{aligned} P(|X| \leq 10) &= P(-10 \leq X \leq 10) = P\left(-\frac{10}{\sigma} \leq \frac{X}{\sigma} \leq \frac{10}{\sigma}\right) \\ &= \Phi\left(\frac{10}{\sigma}\right) - \Phi\left(-\frac{10}{\sigma}\right) \end{aligned}$$

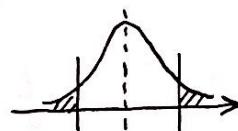
$$\therefore \Phi\left(-\frac{10}{\sigma}\right) = 1 - \Phi\left(\frac{10}{\sigma}\right).$$

$$\therefore \Phi\left(\frac{10}{\sigma}\right) - \Phi\left(-\frac{10}{\sigma}\right) = 2 \cdot \Phi\left(\frac{10}{\sigma}\right) - 1 = 0.1.$$

$$\therefore \Phi\left(\frac{10}{\sigma}\right) = 0.55$$

$$\frac{10}{\sigma} = 0.13$$

$$\sigma = 76.92.$$



↑  
standard Gaussian

**Exercise 3. (FROM FINAL, SPRING 2016)**

A device is deployed in a remote region. The time,  $T$ , to failure, is exponentially distributed with mean 3 years. The device will not be monitored during the first 2 years, so the time before failure can be discovered is  $X = \max(T, 2)$ . What is  $E(X)$ ?

(Hint: Consider separately what happens when  $T < 2$  and  $T \geq 2$ . (That is to say, consider both conditions  $C = \{T < 2\}$  and  $C^c$ . This allows you to apply principles we learned in class, and only do one integration. Without this approach, you may find the following integral (without limits) helpful.)

$$\int xe^{ax} dx = \left( \frac{x}{a} - \frac{1}{a^2} \right) e^{ax}$$

$T \sim \text{exponential w/ mean 3} \rightarrow f_T(t) = \frac{1}{3} e^{-\frac{t}{3}}$  for  $t \geq 0$ .

Method 1 consider  $X = g(T) = \max(T, 2) = \begin{cases} T & T < 2 \\ 2 & T \geq 2 \end{cases}$

$$\begin{aligned} E(X) &= E(g(T)) = \int_{-\infty}^{\infty} g(t) \cdot f_T(t) dt \\ &= \int_0^2 2 \cdot \frac{1}{3} e^{-\frac{t}{3}} dt + \int_2^{\infty} t \cdot \frac{1}{3} e^{-\frac{t}{3}} dt \\ &= 2 + 3e^{-2/3}. \end{aligned}$$

Method 2

$$\begin{cases} E(X|T < 2) = 2 \\ E(X|T \geq 2) = 2 + E(T) = 5 \quad (\text{memoryless property}) \end{cases}$$

$$\begin{aligned} \therefore E(X) &= P(T < 2) \cdot E(X|T < 2) + P(T \geq 2) \cdot E(X|T \geq 2) \\ &= (1 - e^{-2/3}) \cdot 2 + 5e^{-2/3} \\ &= 2 + 3e^{-2/3}. \end{aligned}$$

**Exercise 4.**

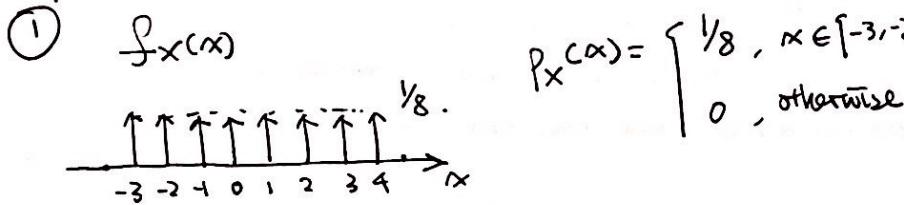
Let random variable  $X$  have pdf  $f_X(x) = 1/(2x^2)$  for  $|x| \geq 1$  and  $f_X(x) = 0$  for  $|x| < 1$ . Let  $Y = \sqrt{|X|}$ . Find  $E(Y)$ .

$$f_X(x) = \begin{cases} \frac{1}{2x^2}, & x \geq 1 \\ 0, & -1 < x < 1 \\ \frac{1}{2x^2}, & x \leq -1 \end{cases}$$

$$\begin{aligned} E(Y) &= E(\sqrt{|X|}) = \int_{-\infty}^{\infty} \sqrt{|x|} \cdot f_X(x) dx \\ &= \int_{-\infty}^{-1} \sqrt{|x|} \cdot \frac{1}{2x^2} dx + \int_{-1}^{1} \sqrt{|x|} \cdot 0 dx + \int_{1}^{\infty} \sqrt{|x|} \cdot \frac{1}{2x^2} dx \\ &= \int_{-\infty}^{-1} (-x)^{\frac{1}{2}} \cdot \frac{1}{2x^2} dx + \int_{1}^{\infty} x^{\frac{1}{2}} \cdot \frac{1}{2x^2} dx \\ &= 2 \cdot \int_{1}^{\infty} x^{\frac{1}{2}} \cdot \frac{1}{2x^2} dx \\ &= 2 \cdot \frac{1}{2} \cdot \int_{1}^{\infty} x^{-\frac{3}{2}} dx \\ &= 2 \end{aligned}$$

**Exercise 5. FROM TEXTBOOK, PROBLEM 4.27 (A-C)**

A voltage  $X$  is uniformly distributed in the set  $\{-3, -2, \dots, 3, 4\}$ . Find the pdf and cdf of the random variables  $X$ ,  $Y = -2X^2 + 3$ ,  $W = \cos(\pi X/8)$ .



$$F_X(x) = \begin{cases} 0, & x < -3 \\ 1/8, & -3 \leq x < -2 \\ 2/8, & -2 \leq x < -1 \\ 3/8, & -1 \leq x < 0 \\ 4/8, & 0 \leq x < 1 \\ 5/8, & 1 \leq x < 2 \\ 6/8, & 2 \leq x < 3 \\ 7/8, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

②  $X: -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$Y = -2X^2 + 3: -15 \quad -5 \quad 1 \quad 3 \quad 1 \quad -5 \quad -15 \quad -29$$

$$\therefore P_Y(y) = \begin{cases} 1/8, & y \in \{3, -29\} \\ 2/8, & y \in \{-15, -5, 1\} \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & y < -29 \\ 1/8, & -29 \leq y < -15 \\ 3/8, & -15 \leq y < -5 \\ 5/8, & -5 \leq y < 1 \\ 7/8, & 1 \leq y < 3 \\ 1, & y \geq 3 \end{cases}$$

③  $X: -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$W = \cos \frac{\pi X}{8}: \cos\left(\frac{3\pi}{8}\right) \cos\left(\frac{7\pi}{8}\right) \cos\frac{\pi}{8} \quad 1 \quad \cos\frac{\pi}{8} \quad \cos\frac{2\pi}{8} \quad \cos\frac{3\pi}{8} \quad \cos\frac{4\pi}{8}.$$

$$\therefore P_W(w) = \begin{cases} 1/8, & w \in \{\cos\frac{7\pi}{8}, 1\} \\ 2/8, & w \in \{\cos\frac{\pi}{8}, \cos\frac{2\pi}{8}, \cos\frac{3\pi}{8}\} \\ 0, & \text{otherwise} \end{cases}$$

$$F_W(w) = \begin{cases} 0, & w < 0 \\ 1/8, & 0 \leq w < \cos\frac{3\pi}{8} \\ 3/8, & \cos\frac{3\pi}{8} \leq w < \cos\frac{2\pi}{8} \\ 5/8, & \cos\frac{2\pi}{8} \leq w < \cos\frac{\pi}{8} \\ 7/8, & \cos\frac{\pi}{8} \leq w < 1 \\ 1, & w \geq 1 \end{cases}$$

## alternative expressions .

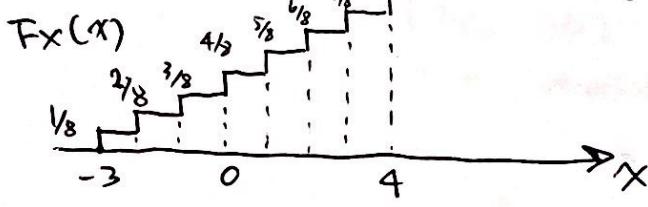
Exercise 5. FROM TEXTBOOK, PROBLEM 4.27 (A-C)

A voltage  $X$  is uniformly distributed in the set  $\{-3, -2, \dots, 3, 4\}$ . Find the pdf and cdf of the random variables  $X$ ,  $Y = -2X^2 + 3$ ,  $W = \cos(\pi X/8)$ .

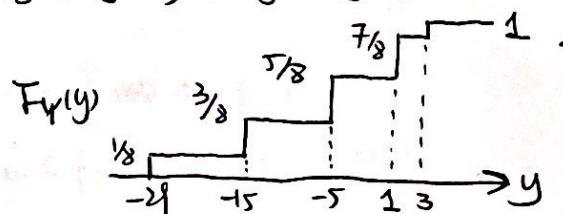
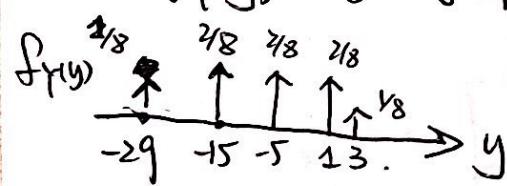
$$\textcircled{1} \quad f_X(x) = \frac{1}{8} \cdot (\delta(x+3) + \delta(x+2) + \delta(x+1) + \delta(x) + \delta(x-1) + \delta(x-2) + \delta(x-3) + \delta(x-4))$$



$$F_X(x) = \int_{-\infty}^x f_X(x) dx = \frac{1}{8} (u(x+3) + u(x+2) + u(x+1) + u(x) + u(x-1) + u(x-2) + u(x-3) + u(x-4))$$



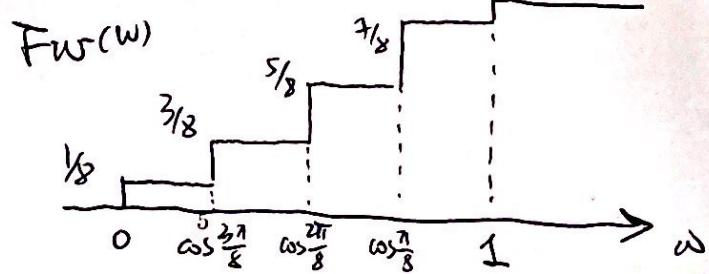
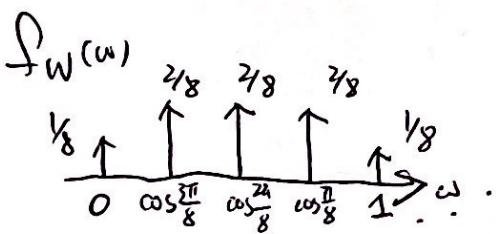
$$\textcircled{2} \quad f_Y(y) = \frac{1}{8} \cdot \delta(y+29) + \frac{2}{8} \cdot \delta(y+15) + \frac{2}{8} \cdot \delta(y+5) + \frac{2}{8} \cdot \delta(y-1) + \frac{1}{8} \cdot \delta(y-3)$$



$$F_Y(y) = \frac{1}{8} u(y+29) + \frac{2}{8} u(y+15) + \frac{2}{8} u(y+5) + \frac{2}{8} u(y-1) + \frac{1}{8} u(y-3)$$

$$\textcircled{3} \quad f_W(w) = \frac{1}{8} \delta(w) + \frac{2}{8} \delta(w - \cos \frac{3\pi}{8}) + \frac{2}{8} \delta(w - \cos \frac{2\pi}{8}) + \frac{2}{8} \delta(w - \cos \frac{\pi}{8}) + \frac{1}{8} \delta(w-1)$$

$$F_W(w) = \frac{1}{8} u(w) + \frac{3}{8} u(w - \cos \frac{3\pi}{8}) + \frac{2}{8} u(w - \cos \frac{2\pi}{8}) + \frac{2}{8} u(w - \cos \frac{\pi}{8}) + \frac{1}{8} u(w-1)$$



**Exercise 6. (FROM TEXTBOOK, PROBLEM 4.91)**

Let  $Y = \exp(X)$ .

- Find the cdf of  $Y$  in terms of the cdf of  $X$ .
- Find the pdf of  $Y$  in terms of the pdf of  $X$ .
- Find the pdf of  $Y$  for the specific case when  $X$  is Gaussian. In this case,  $Y$  is a log-normal RV.

a)  $F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(y))$   
 $= F_X(\ln(y))$

b)  $f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(\ln(y)) \cdot \frac{1}{y}$

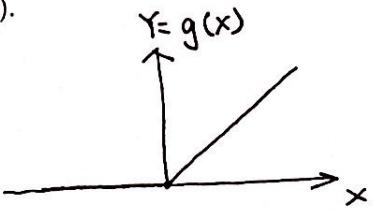
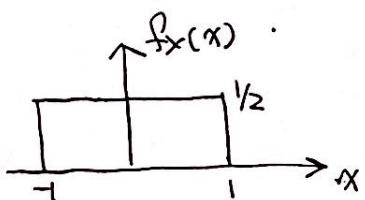
c)  $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$   
 $\therefore f_Y(y) = \frac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}$

**Exercise 7. (EXAM 3 FALL 2016)**

Let  $X$  be a voltage input to a rectifier, and let  $X$  be a continuous uniform random variable on the interval  $[-1, 1]$ . The rectifier output is a random variable  $Y$ , where

$$Y = g(X) = \begin{cases} 0 & X < 0 \\ X & X \geq 0 \end{cases}$$

Find and sketch the PDF of  $Y$ ,  $f_Y(y)$ .



For  $y < 0$ :  $F_Y(y) = 0$ .

$$\begin{aligned} y = 0: \quad F_Y(y) &= P(Y \leq 0) = P(g(X) \leq 0) = P(X \leq 0) \\ &= \int_{-\infty}^0 f_X(x) dx = \int_{-1}^0 \frac{1}{2} dx = \frac{1}{2}. \end{aligned}$$

$$y > 0: \quad F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y)$$

now take the derivative to get:

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{2} \delta(y), & y = 0 \\ \frac{1}{2}, & 0 < y \leq 1 \\ 0, & y > 1. \end{cases}$$