

Exercise 1. (FROM TEXTBOOK, PROBLEMS 4.30, 4.32)
THIS PROBLEM IS WORTH 2 POINTS

A random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - \frac{1}{4}e^{-2x} & \text{for } x \geq 0 \end{cases}$$

NOTE! This is the same CDF as an earlier problem.

- (a) Find $F_X(x|A)$, where $A = \{X > 0\}$.
- (b) Find $F_X(x|C)$, where $C = \{X = 0\}$
- (c) Find $f_X(x|B)$ and $F_X(x|B)$, where $B = \{X > 0.25\}$

a.) $F_X(x|A) = P(X \leq x | X > 0) = \frac{P(\{X \leq x\} \cap \{X > 0\})}{P(X > 0)}.$

$$P(X > 0) = 1 - P(X \leq 0) = 1 - \frac{3}{4} = \frac{1}{4}.$$

$$P(\{X \leq x\} \cap \{X > 0\}) = \begin{cases} F_X(x) - P(X=0) = 1 - \frac{1}{4}e^{-2x} - \frac{3}{4}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$\therefore F_X(x|A) = \begin{cases} 1 - e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0. \end{cases}$$

b.) Similarly, $F_X(x|C) = \frac{P(\{X \leq x\} \cap \{X=0\})}{P(X=0)}.$

$$= \begin{cases} \frac{P(X=0)}{P(X=0)} = 1, & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

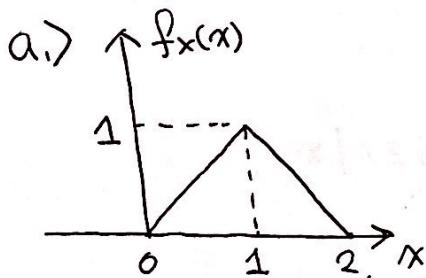
c.) $F_X(x|B) = \frac{P(\{X \leq x\} \cap \{X > 0.25\})}{P(X > 0.25)} = \begin{cases} \frac{F_X(x) - F_X(0.25)}{1 - F_X(0.25)} = \frac{\frac{1}{4}e^{-2x} - \frac{1}{4}}{e^{-1/2}}, & x > 0.25 \\ 0 & x \leq 0.25. \end{cases}$

$$f_X(x|B) = \frac{d}{dx} F_X(x|B) = \begin{cases} \frac{2e^{-2x}}{e^{-1/2}}, & x > 0.25 \\ 0, & x \leq 0.25. \end{cases}$$

Exercise 2. (FROM EXAM 2 OF 3; SPRING 2016))
Consider the random variable X with PDF given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Sketch $f_X(x)$. Label axes and relevant values.
 (b) Find and sketch the conditional density $f_{X|A}(x|A)$ for the event $A = \{X < 1/4\}$.
 (c) What is the conditional mean $E(X|B)$ for the event $B = \{2/3 < X < 4/3\}$?
 (Hint: you do not need to find $P(B)$ to solve part (c)!)

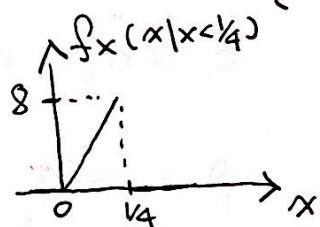


b.) $P(A) = P(X < 1/4) = \int_0^{1/4} x dx$

$$= \frac{x^2}{2} \Big|_0^{1/4} = \frac{1}{32}.$$

$$f_{X|A}(x|A) = \begin{cases} \frac{f(x)}{P(A)}, & \text{for } 0 < x < 1/4 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 32x & \text{for } 0 < x < 1/4 \\ 0 & \text{otherwise} \end{cases}$$



c.) $f_{X|B}(x|B) = \begin{cases} \frac{x}{P(B)}, & \text{for } 2/3 < x \leq 1 \\ \frac{2-x}{P(B)}, & \text{for } 1 < x < 4/3 \\ 0, & \text{otherwise} \end{cases}$

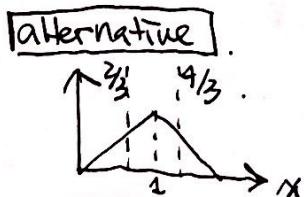
$$P(B) = \int_{2/3}^{4/3} f_X(x) dx$$

$$= \int_{2/3}^1 x dx + \int_1^{4/3} 2-x dx$$

$$= 5/18 + 5/18 = 5/9$$

$$\therefore E(X|B) = \int_{-\infty}^{\infty} f_{X|B}(x|B) x dx = \int_{2/3}^1 x \cdot \frac{x}{5/9} dx + \int_1^{4/3} x \cdot \frac{2-x}{5/9} dx$$

$$= 19/45 + 26/45 = 1.$$



Since $f_{X|B}(x|B) = \begin{cases} x/P(B), & 2/3 < x \leq 1 \\ 2-x/P(B), & 1 < x < 4/3 \end{cases}$, it is symmetric about $x=1$.

$$\therefore E(X|B) = 1.$$

Exercise 3.

THIS PROBLEM IS WORTH 2 POINTS

Let X be a uniform random variable on the interval $(0, 10)$.

$$f_X(x) = \frac{1}{10} \quad \text{for } x \in (0, 10).$$

- (a) Find $P(X \leq 6)$ and $P(X > 8)$.
- (b) Compute the conditional PDF's of $f_X(x|X \leq 6)$ and $f_X(x|X > 8)$.
- (c) Find the conditional means of $E(X|X \leq 6)$ and $E(X|X > 8)$.
- (d) Find the conditional variances of $Var(X|X \leq 6)$ and $Var(X|X > 8)$.

$$\text{a.) } P(x \leq 6) = \int_0^6 \frac{1}{10} dx = \frac{6}{10}. \quad P(x > 8) = 1 - P(x \leq 8) \\ = 1 - \frac{8}{10} = \frac{2}{10}.$$

$$\text{b.) } f_{X|x}(x|x \leq 6) = \begin{cases} \frac{f(x)}{P(x \leq 6)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}, & \text{for } 0 < x \leq 6 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{X|x}(x|x > 8) = \begin{cases} \frac{f(x)}{P(x > 8)} = \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{2}, & \text{for } 8 < x < 10 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{c.) } E(x|x \leq 6) = \int_0^6 x \cdot \frac{1}{6} dx = 3$$

$$E(x|x > 8) = \int_8^{10} x \cdot \frac{1}{2} dx = 9$$

$$\text{d.) } E(x^2|x \leq 6) = \int_0^6 x^2 \cdot \frac{1}{6} dx = 12.$$

$$\text{var}(x|x \leq 6) = E(x^2|x \leq 6) - (E(x|x \leq 6))^2 \\ = 12 - 9 = 3.$$

$$E(x^2|x > 8) = \int_8^{10} x^2 \cdot \frac{1}{2} dx = 244/3$$

$$\text{var}(x|x > 8) = E(x^2|x > 8) - (E(x|x > 8))^2 \\ = 244/3 - 9^2 = 1/3.$$