Name: ______________________ PU ID: ______________

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Homework 4

Fall 2021

(Due Thursday September 23 at 11:59pm)

Homework is due on **Thursday September 23 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

Statement:

Topics: Moments and their properties (Section 3.1-3.3, Sections 4.1-4.3, except Section 4.2.2)); common PMFs (Chapter 3.5)

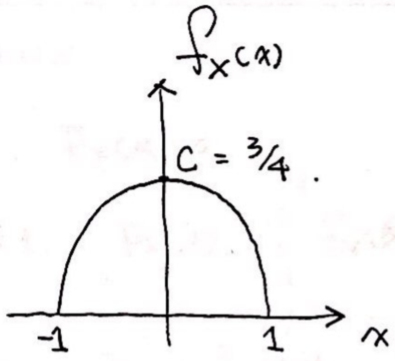
Exercise 1. (FROM TEXTBOOK, PROBLEMS 4.17, 4.39)

A random variable X has pdf:

$$f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c and sketch the pdf.

(a) Sketch:



For $f_X(x)$ to be a valid pdf, we need

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

$$\therefore \int_{-1}^{1} c\cdot(1-x^2) = 1$$

$$c\left(x - \tfrac{1}{3}x^3\right)\Big|_{-1}^{1} = 1.$$

$$\therefore c \cdot 4/3 = 1$$

$$c = 3/4$$

(a) $c =$ $3/4$

(b) Find and sketch the cdf of X.

for $x < -1$. $F_X(x) = 0$.

for $-1 \leq x \leq 1$, $F_X(x) = \int_{-1}^{x} f_X(s)\,ds$

$$= \int_{-1}^{x} 3/4\,(1-s^2)\,ds$$

$$= 3/4\left(s - \tfrac{1}{3}s^3\right)\Big|_{-1}^{x}$$

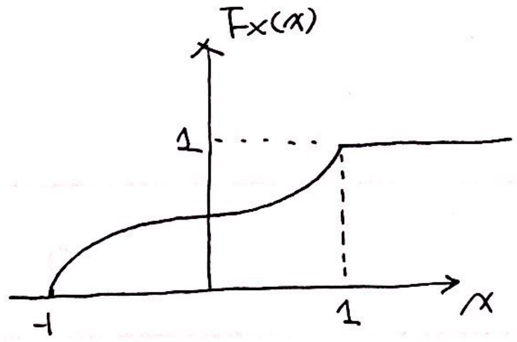
$$= 3/4\,x - 1/4\cdot x^3 + \tfrac{1}{2}.$$

for $x > 1$, $F_X(x) = 1$.

(b) CDF:

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 3/4\,x - 1/4\,x^3 + 1/2, & -1 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

(b) Sketch:



(c) Use the cdf to find $P(X = 0)$, $P(0 < X < 0.5)$, $P(|X - 0.5| < 0.25)$.

$$P(X=0) = F_X(0^+) - F_X(0^-) = 0.$$

$$P(0 < X < 0.5) = F_X(0.5) - F_X(0) = \frac{27}{32} - \frac{1}{2}$$

$$= \frac{11}{32}.$$

$$P(|X - 0.5| < 0.25) = P(0.25 < X < 0.75)$$

$$= F_X(0.75) - F_X(0.25)$$

$$= \frac{245}{256} - \frac{175}{256}$$

$$= \frac{70}{256}.$$

$P(X = 0) =$ 0

$P(0 < X < 0.5) =$ $11/32$

$P(|X - 0.5| < 0.25) =$ $70/256$.

Exercise 2. (FROM TEXTBOOK, PROBLEMS 4.17, 4.39)
A random variable X has pdf:

$$f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

NOTE: This problem is worth half a point. This is the same CDF as Exercise 1 above.

Find the mean and variance of X.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx$$

$$= \int_{-1}^{1} x \cdot 3/4 \cdot (1-x^2)\, dx$$

$$= 3/8 \cdot x^2 - 3/16 \cdot x^4 \Big|_{-1}^{1}$$

$$= 0 .$$

$$\text{var}(X) = E(x^2) - E^2(x)$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_{-1}^{1} x^2 \cdot 3/4\, (1-x^2)\, dx$$

$$= 1/4 \cdot x^3 - 3/20 \cdot x^5 \Big|_{-1}^{1} = 1/5 .$$

$$\therefore \text{var}(x) = 1/5 - 0^2 = 1/5 .$$

Mean:	0

Variance:	$1/5$.

Exercise 3. (FROM TEXTBOOK, PROBLEMS 4.13, 4.23, PLUS)

A random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - \frac{1}{4}e^{-2x} & \text{for } x \geq 0 \end{cases}$$

NOTE: This problem is worth half a point. This is the same CDF as HW3 Fall 2021 Exercise 4. You may find the associated PDF for this RV in the posted solutions for HW3.

Find mean and variance of X.

From last homework we have its pdf as

$$f_X(x) = \begin{cases} 0, & x<0 \\ 3/4\,\delta(x), & x=0 \\ 1/2\, e^{-2x}, & x>0. \end{cases}$$

$$\therefore E(x) = \int_0^\infty x \cdot 1/2\, e^{-2x}\, dx + \overbrace{0 \cdot P(x=0)}^{\int 3/4\,\delta(x)\cdot x\, dx}.$$

$$= \left(-\frac{1}{4}x\cdot e^{-2x} - \frac{1}{8}\cdot e^{-2x}\right)\Big|_0^\infty + 0 \cdot 3/4$$

$$= 1/8$$

$$E(x^2) = \int_0^\infty x^2 \cdot 1/2\, e^{-2x} dx + \int 3/4\,\delta(x)\cdot x^2\, dx.$$

$$= 1/8 + 0^2 \cdot 3/4 = 1/8$$

$$\therefore var(x) = E(x^2) - E^2(x) = 1/8 - 1/64$$

$$= 7/64$$

Mean:	1/8

Variance:	7/64

Exercise 4.
A telephone installation has 2 lines, which allows zero, one, or two calls to happen simultaneously. The pmf of N, the number of active calls, is

$$P_N(n) = \begin{cases} 0.2 & n = 0 \\ 0.7 & n = 1 \\ 0.1 & n = 2 \\ 0 & otherwise \end{cases}$$

(a) Find $E(N)$, the expected number of calls active.

$E(N) = \sum_{n \in \{0,1,2\}} n \cdot P(N=n) = 0{,}2 \cdot 0 + 0{,}7 \cdot 1 + 0{,}1 \cdot 2.$
$= 0{,}9$

(a): $E(N) = 0{,}9$

(b) Find $E(N^2)$, the second moment of N.

$E(N^2) = \sum_n n^2 \cdot P(N=n) = 0{,}2 \cdot 0^2 + 0{,}7 \cdot 1^2 + 0{,}1 \cdot 2^2$
$= 1{,}1$

(b): $E(N^2) = 1{,}1$

(c) Find $Var(N)$, the variance of N

$var(N) = E(N^2) - E^2(N) = 1{,}1 - 0{,}9^2$
$= 0{,}29$

(c): $var(N) = 0{,}29$

(d) Find σ_N, the standard deviation of N.

$\sigma(N) = \sqrt{var(N)} = \sqrt{0{,}29} \approx 0{,}539$

(d): $\sigma(N) \approx 0{,}539$

Exercise 5. (FROM EXAM 1, SPRING 2019)

Suppose a certain professor gives a very hard exam, and would like to curve the scores to obtain a higher average. Let X be the random variable indicating the initial grades, and $Y = aX + b$ be the curved grades, where a and b are constants. She would also like to preserve the *order* of the grades, so that the highest score X is still the highest score Y.

If $E(X) = 20$ and $Var(X) = 50$, what values of a and b should she pick to obtain $E(Y) = 75$ and $Var(Y) = 800$?

$E(Y) = E(\alpha X + b) = \alpha \cdot E(X) + b.$

plug in the desired value:

$75 = \alpha \cdot 20 + b.$ ①.

$var(Y) = var(\alpha X + b) = \alpha^2 \cdot var(X)$

plug in the desired value:

$800 = \alpha^2 \cdot 50$ ②

$\therefore \begin{cases} 75 = \alpha \cdot 20 + b \\ 800 = \alpha^2 \cdot 50 \end{cases} \rightarrow$ solve for $\begin{cases} \alpha_1 = 4 \\ b_1 = -5. \end{cases}$ $\begin{cases} \alpha_2 = -4 \\ b_2 = 155. \end{cases}$

$\because$ we want to preserve the order,

$\therefore$ α must be positive (otherwise the order would flip)

$\therefore \alpha = 4, b = -5.$

$a =$ 4

$b =$ -5.