Name: ______________________________ PU ID: ____________

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

PURDUE
UNIVERSITY

Homework 11

Fall 2021
(Due Thursday December 2, 11:59pm)

Homework is due on **Thursday December 2 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

Statement:

Topics: Law of Large Numbers and Central Limit Theorem)

Exercise 1. (FROM TEXTBOOK, PROBLEM 7.16, REPHRASED SOMEWHAT)
Suppose that 20% of voters are in favor of a certain legislation. A large number n of voters are polled and the proportion in favor is estimated using a relative frequency. That is to say, the fraction of the n voters polled who are in favor is the estimate for the overall population.
Use the Chebyshev inequality associated with the Law of Large Numbers to approximate how many voters should be polled so that the probability is at least 0.95 that the estimate differs from 0.20 by less than 0.02.

$\because X \sim$ Bernoulli w/ $p=0.2$.

$\therefore \sigma^2 = p(1-p) = 0.16$.

P(estimation differ from 0.2 by less than 0.02) ≥ 0.95

actual mean

$\Rightarrow P(|M_n - \mu| < \underbrace{0.02}_{\varepsilon}) \geq 1 - \frac{\sigma^2}{n \cdot \varepsilon^2} = 0.95$

$1 - \frac{0.16}{n \cdot 0.02^2} = 0.95$

$\therefore n = 8000$.

Exercise 2. (FROM TEXTBOOK, PROBLEM 7.23)
(NOTE: This is a different problem than above.) A large number n of voters are polled and the proportion in favor is estimated using a relative frequency. That is to say, the fraction of the n voters polled who are in favor is the estimate for the overall population.
Use equation the Central Limit Theorem to determine how many voters should be polled so that the probability is at least 0.95 that $f_A(n)$ differs from 0.20 by less than 0.02.

$$Z_n = \frac{\sqrt{n}}{\sigma}(M_n - \mu)$$

we want $P(0.18 \le M_n \le 0.22)$

$$= P\left(\frac{\sqrt{n}}{\sigma}(0.18-0.2) \le Z_n \le \frac{\sqrt{n}}{\sigma}(0.22-0.2)\right) = 0.95.$$

by checking the standard Gaussian table, we can find that $P(-1.96 \le Z_n \le 1.96) = 0.95$

$$\therefore \frac{\sqrt{n}}{\sigma}\cdot(0.18-0.2) = -1.96$$

$\because X \sim$ Bernoulli w/ $p=0.2$

$$\therefore \sigma^2 = p\cdot(1-p) = 0.16.$$

$$\sigma = \sqrt{0.16} = 0.4.$$

$$\therefore \frac{\sqrt{n}}{0.4}(0.18-0.2) = -1.96.$$

$$\sqrt{n} = 39.2$$

$$n \approx 1537.$$

Exercise 3.
If 10 fair dice are thrown, estimate the probability (using the Central Limit Theorem) that the sum obtained is between 30 and 40 (inclusive).

central limit theorem :

$X \sim$ uniform $\in [1,2,3,4,5,6]$.

$\therefore \mu = 3.5, \quad \sigma^2 = \frac{35}{12}$.

$Z_n = \frac{S_n - n \cdot \mu}{\sigma \cdot \sqrt{n}}$. (sum)

$Z_n = \frac{\sqrt{n}\,(M_n - \mu)}{\sigma}$ (mean)

$$P(30 \leq S_n \leq 40) = P\left(\frac{30 - 10 \cdot 3.5}{\sqrt{\frac{35}{12}} \cdot \sqrt{10}} \leq Z_n \leq \frac{40 - 10 \cdot 3.5}{\sqrt{\frac{35}{12}} \cdot \sqrt{10}}\right)$$

$$\text{(alternatively)} = P\left(\frac{3 - 3.5}{\sqrt{\frac{35}{10}}} \cdot \sqrt{10} \leq Z_n \leq \frac{\sqrt{10}\,(4 - 3.5)}{\sqrt{\frac{35}{12}}}\right)$$

$$= P(-0.926 \leq Z_n \leq 0.926)$$

$$= 0.646.$$

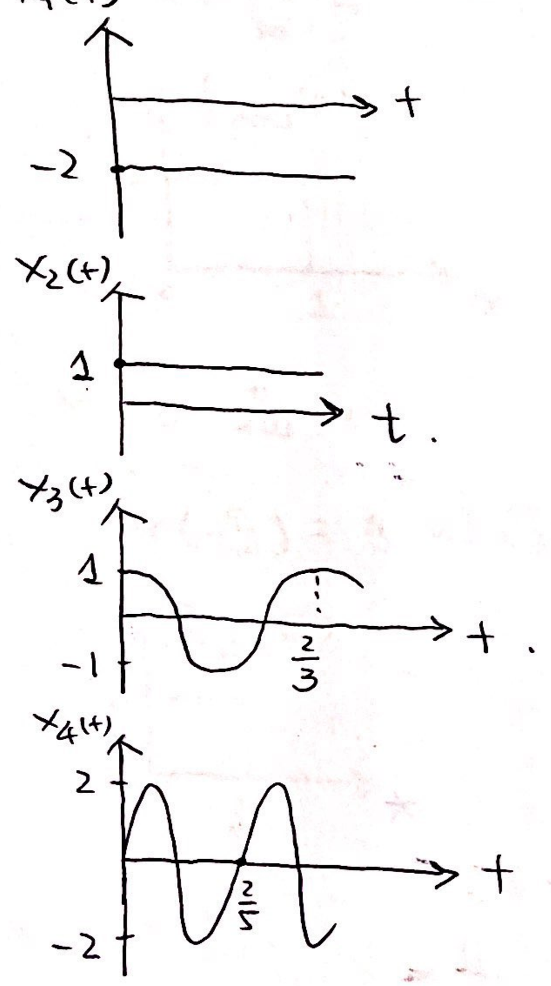
Exercise 4.
Suppose a random process has 4 equally likely sample functions (i.e., realizations), for $t > 0$, given by:

$$\begin{aligned} x_1(t) &= -2 \\ x_2(t) &= 1 \end{aligned}$$

$$x_3(t) = \cos 3\pi t$$

$$x_4(t) = 2 \sin 5\pi t$$

Sketch the ensemble (i.e., the set of realizations). What is the sample space if $t = .5$?



at $t = 0.5$.

$x_1(0.5) = -2$,

$x_2(0.5) = 1$

$x_3(0.5) = 0$

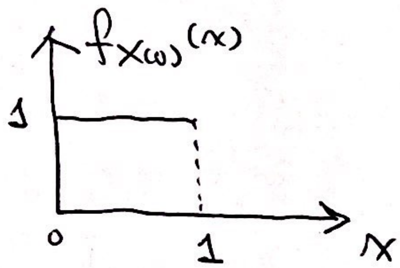
$x_4(0.5) = 2$.

$\therefore S_{X(0.5)} = \{-2, 0, 1, 2\}$.

Exercise 5.
Consider $X(t) = A\cos\omega t$, where A is a random variable that is uniform on the interval $[0,1]$. Determine the pdf's of $X(t)$ when $t = 0, \pi/4\omega, \pi/2\omega$, and π/ω.

at $t=0$:

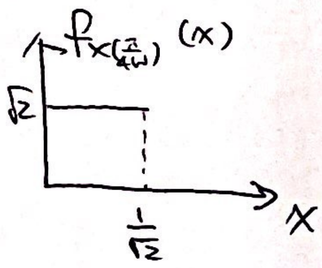
$X(t)\big|_{t=0} = A \qquad \therefore X(t)\big|_{t=0} \sim \text{uniform } (0,1).$



$$f_{X(0)}(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

at $t = \frac{\pi}{4\omega}$:

$X(\frac{\pi}{4\omega}) = A \cdot \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \cdot A. \quad \sim \text{uniform } (0, \frac{1}{\sqrt{2}}).$

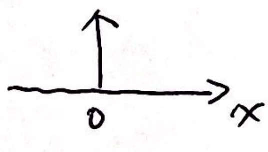


$$f_{X(\frac{\pi}{4\omega})}(x) = \begin{cases} \sqrt{2}, & 0 \le x \le \frac{1}{\sqrt{2}} \\ 0, & \text{otherwise.} \end{cases}$$

at $t = \frac{\pi}{2\omega}$:

$\delta(x)$.

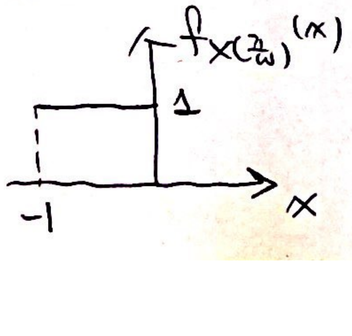
$X(\frac{\pi}{2\omega}) = 0.$



$f_{X(\frac{\pi}{2\omega})}(x) = \delta(x).$

at $t = \frac{\pi}{\omega}$:

$X(\frac{\pi}{\omega}) = A \cdot \cos(\pi) = -A \quad \sim \text{uniform } (-1, 0)$



$$f_{X(\frac{\pi}{\omega})}(x) = \begin{cases} 1, & -1 \le x \le 0 \\ 0, & \text{otherwise.} \end{cases}$$