Name: ______________________ PU ID: ____________

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Homework 10

Fall 2021
(Due Thursday November 18, 11:59pm)

Homework is due on **Thursday November 18 at 11:59pm** on Gradescope. No late homework will be accepted, and no homework will be accepted without a statement. Include a brief description of all sources of information you used (including other people), not counting the text, handouts, or material posted on the web page, **or** state "I did not receive help on this homework". You do not need to reference any material presented in class or on the course web-site, in the textbook, nor Prof. Reibman nor TA Haoyu Chen.

Statement:

Topics: Joint moments (Chapter 5.6); Conditional Probability (Chapter 5.7)

Exercise 1. (FROM TEXTBOOK, PROBLEMS 5.26, 5.65, AND 5.80)
Let X and Y have the joint pdf

$$f_{X,Y}(x,y) = (x+y), \text{ for } 0 \le x \le 1, 0 \le y \le 1$$

From the last homework, we know that the marginal PDF's are

$$f_X(x) = (x + 1/2), \text{ for } 0 \le x \le 1$$

$$E(X) = \int_0^1 x\cdot(x+\tfrac{1}{2})\,dx$$

$$f_Y(y) = (1/2 + y), \text{ for } 0 \le y \le 1$$

$$= 7/12 = E(Y).$$

(a) Find the correlation and covariance of X and Y.

(b) Determine if X and Y are independent, orthogonal, or uncorrelated. How do you know?

correlation

a). $E(XY) = \iint_{\mathbb{R}^2} xy\cdot f_{XY}(x,y)\,dxdy = \int_0^1\int_0^1 x\cdot y(x+y)\,dxdy$

$= \int_0^1 \frac{x^3}{3}y + \frac{x^2}{2}y^2 \Big|_0^1 dy = \int_0^1 \frac{y}{3} + \frac{y^2}{2}\,dy = \frac{y^2}{6} + \frac{y^3}{6}\Big|_0^1 = \boxed{1/3.}$

Covariance:

$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y). = 1/3 - (7/12)^2 = \boxed{-\frac{1}{144}.}$

b). $E(XY) \neq E(X)\cdot E(Y)$ — not independent

$E(XY) \neq 0$ — not orthogonal

$\mathrm{cov}(XY) \neq 0$ — not uncorrelated.

Exercise 2. (FROM TEXTBOOK, PROBLEMS 5.26, 5.65, AND 5.80)
Using the same joint and marginal PDFs from the previous exercise:

(c) Find the conditional pdf of Y given $X = x$, namely, $f_Y(y|x)$.

(d) Find $E(Y|X = x)$. (You can use the result from (c).)

c.) $f_Y(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{x+y}{x+1/2}$. for $x \in [0,1]$
$y \in [0,1]$.

d.) $E(Y|X=x) = \displaystyle\int_0^1 y \cdot \frac{x+y}{x+1/2}\, dy$

$= \dfrac{1}{x+1/2} \cdot \displaystyle\int_0^1 xy + y^2\, dy$

$= \dfrac{3x+2}{6x+3}$.

Exercise 3. (Exam 3, Fall 2016) (cut here into 2 problems)
Given X with PDF

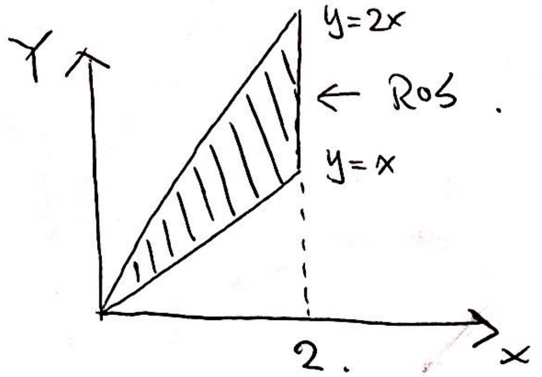
$$f_X(x) = \begin{cases} 1/2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then Y is uniformly distributed between x and $2x$.

(a) Sketch the region of support. That is, indicate where $f_{XY}(x, y)$ is nonzero.

(b) Find the joint PDF $f_{XY}(x, y)$.

a)



b.) . $f_X(x) = \frac{1}{2}$ for $0 \le x \le 2$.

$f_Y(y|X=x) = \frac{1}{2x-x} = \frac{1}{x}$ for $x \le y \le 2x$.

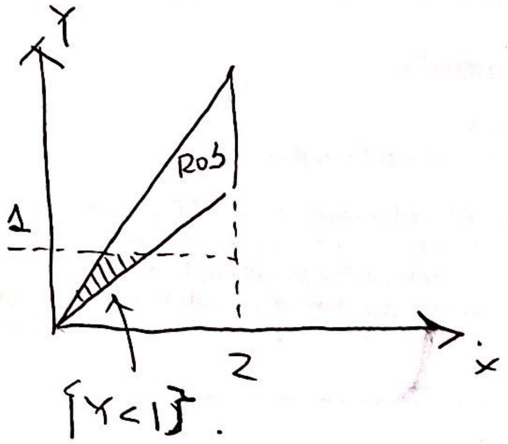
$\therefore f_{XY}(x,y) = f_X(x) \cdot f_Y(y|x) = \begin{cases} \frac{1}{2x}, & \text{for } 0 \le x \le 2,\ x \le y \le 2x. \\ 0, & \text{otherwise.} \end{cases}$

Exercise 4. (EXAM 3, FALL 2016) (THE SECOND PART)
Using the same conditional, marginal, and joint PDFs from the previous exercise:

(c) What is $P(Y < 1)$?

(d) What is $E(Y)$? (Hint: you may use the law of iterated expectations.)

c.).



$$P(Y<1) = \int_0^1 \int_{y/2}^{y} f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^1 \frac{1}{2}\left(\ln(y) - \ln(y/2)\right) dy$$

$$= \int_0^1 \frac{1}{2}\left(\ln(y) - \ln(y) + \ln(2)\right) dy$$

$$\boxed{= \frac{1}{2}\ln(2).}$$

d.) iterated expectation

$$E(Y) = E_X\left(E(Y|X)\right)$$

$$= E_X\left(\int_x^{2x} y \cdot f_Y(y|x)\, dy\right)$$

$$= E_X\left(\frac{3x}{2}\right)$$

$$= \int_0^1 \frac{3x}{2} \cdot f_X(x)\, dx$$

$$= \int_0^1 \frac{3x}{2} \cdot \frac{1}{2}\, dx$$

$$\boxed{= 3/2.}$$