FINAL EXAM: Friday 8-10am

Spring 2018, MWF 2:30-3:20
(May 4, 2018)

This is a closed book exam. There are 4 YES/NO problems, 3 multiple choice problems, and 6 workout problems.

Neither calculators nor help sheets are allowed. Some formulae and helpful information is attached at the end.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name on every page of the exam and turn in everything when time is up.

Name: ______________________ PUID: ______________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

For each of the YES/NO problems, mark you answer CLEARLY.
I cannot give credit for unclear answers.

Problem 1. (YES/NO: 5 POINTS)

If $X(t)$, a WSS random process, is input to a linear time-invariant system, then $E(Y(t)) = H(0)E(X(t))$.

Problem 2. (YES/NO: 5 POINTS)

For arbitrary events A and B, $P(A \cap B) + P(A \cup B) = 2P(A) + 2P(B)$.

Problem 3. (YES/NO: 5 POINTS)

If X is a continuous random variable with a PDF that is symmetric about 0, then
$P(X > a) = F_x(-a)$ for all a.

Problem 4. (YES/NO: 5 POINTS)

If events A and B are independent, then $P(A \cup B) = P(A) + P(B)$.

Problem 5. (MULTIPLE CHOICE: 5 POINTS)

For two random variables X and Y, $VAR(2X + 3Y + 4)$ can be computed as

(a) $2\ VAR(X) + 3\ VAR(Y) + 4$

(b) $4\ VAR(X) + 9\ VAR(Y)$

(c) $2\ VAR(X) + 3\ VAR(Y)$

(d) $4\ VAR(X) + 9\ VAR(Y) + 4$

(e) $4\ VAR(X) + 9\ VAR(Y) + 12\ COV(X, Y)$

(f) $4\ VAR(X) + 9\ VAR(Y) + 6\ COV(X, Y)$

(g) $4\ VAR(X) + 9\ VAR(Y) - 12\ COV(X, Y)$

(h) None of the above

(i) Too little information to solve.

Problem 6. (MULTIPLE CHOICE: 5 POINTS)

The Law of Large Numbers bounds the probability that the sample mean M_n of a random variable X varies from the true mean of X by more than ϵ when the number of samples n is large. Specifically,

$$P(|M_n - \mu| \geq \epsilon) \leq \sigma^2/n\epsilon^2.$$

Suppose student heights range from 120 to 220 cm, but we don't know the average. We plan to estimate the average student height using the sample mean. How many students' heights must we measure to ensure the sample mean is within 0.25 cm of the true average height with probability at least 0.9?

Assume the heights are uncorrelated and have unit variance. For a given ϵ and σ^2, how large should n be to ensure that the sample mean M_n is no more than ϵ away from μ, the true mean of X?

(a) 4.4

(b) 17.8

(c) 40

(d) 160

(e) 800

(r) None of the above.

(g) Too little information to solve.

Problem 7. (5 POINTS)

Blocks on a computer disk are good with probability p and bad with probability $1 - p$. Blocks are good or bad independently of each other. Let X be the location (starting from 1) of the first bad block. What is the PMF of X?

Problem 8. (10 POINTS)

Bob wants to buy a wearable camera. He finds 3 cameras whose prices are appropriate for him: a Gopro, a Qlippie, and a Pivothead. The Qlippie can only be mounted on the chest, the Pivothead can only be mounted on the head, and the Gopro can be mounted in either place.
Suppose the probability that he buys the Gopro is twice the probability he buys the Pivothead, and the probability he buys the Qlippie is 0.4.
Also, if he buys the GoPro, then the probability that he mounts the camera on his head is 0.6.

(a) What is probability he does **NOT** buy the GoPro to mount on his head?

(b) Given that he mounts his camera on his chest, what is the probability he bought the Qlippie?

Problem 9. (15 POINTS)

Suppose X has the CDF

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(a) What is the PDF of X?

(b) Suppose Y is uniformly distributed between $[X-1/2, X+1/2]$. Find the joint PDF, $f_{XY}(x,y)$, **including its region of support**. (A sketch of the region of support may be helpful.)

(c) What is $E(Y)$? (Hint: It will be much faster to use the theorem of iterated expectations).

Problem 10. (10 POINTS)
Let X be the voltage output from a microphone which is Gaussian with mean 0 and variance 16. Then X is input to a limiter circuit, with cut-off ± 4. Thus the output of the limiter Y is given by

$$Y = g(X) = \begin{cases} -4 & \text{for } X < -4 \\ X & \text{for } -4 \leq X \leq 4 \\ 4 & \text{for } X > 4 \end{cases}$$

Find and sketch the PDF of Y, $f_Y(y)$. Express your answer in terms of the Φ-function.

Problem 11. (15 POINTS)

Let X and Y be independent random variables, with

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $VAR(Y)$

(b) What is the correlation coefficient between X and Y: ρ_{XY}?

(c) Find $P(X < Y)$

Problem 12. (10 POINTS)

Let $X(t) = At^2 + B$ where A and B are independent random variables with means μ_A, μ_B and unit variances.

(a) What is the mean function $E(X(t))$?

(b) Find the auto-covariance function $C_X(t_1, t_2)$.

(c) (Bonus 2 points) Is $X(t)$ Wide Sense Stationary? Give a reason why or why not.

Problem 13. (Multiple choice: 5 points)

Let X and Y be independent random variables, with moment generating functions

$$M_X(s) = 0.2 + 0.2\exp(s) + 0.6\exp(2s)$$

and

$$M_Y(s) = 0.5\exp(s) + 0.5\exp(2s)$$

respectively, for $-\infty < s < \infty$. What is the $E(Z^3)$ where the random variable Z is defined by $Z = X + Y$?

(If you show your work you may receive partial credit.)

(a) 1.0

(b) 2.5

(c) 9.6

(d) 31.7

(e) None of the above

Empty page to show more work.
Label problems clearly!
(I need to be able to find your work to give you credit.)

Discrete Random Variables

- Bernoulli Random Variable, parameter p
  $S = \{0, 1\}$
  $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
  $E(X) = p; \mathrm{VAR}(X) = p(1-p)$
  $M_X(s) = 1 + p + p\exp(s)$

- Binomial Random Variable, parameters (n, p)
  $S = \{0, 1, \ldots, n\}$
  $p_k = \binom{n}{k} p^k (1-p)^{n-k}; k = 0, 1, \ldots, n; 0 \leq p \leq 1$
  $E(X) = np; \mathrm{VAR}(X) = np(1-p)$
  $M_X(s) = (1 + p + p\exp(s))^n$

- Geometric Random Variable, parameter p
  $S = \{0, 1, \ldots\}$
  $p_k = p(1-p)^k; k = 0, 1, \ldots, ; 0 \leq p \leq 1$
  $E(X) = 1/p; \mathrm{VAR}(X) = (1-p)/p^2$
  $M_X(s) = pe^s/(1 - (1-p)e^s)$

- Poisson Random Variable, parameter α
  $S = \{0, 1, \ldots\}$
  $p_k = \alpha^k e^{-\alpha}/k! \quad k = 0, 1, \ldots,$
  $E(X) = \alpha; \mathrm{VAR}(X) = \alpha$
  $M_X(s) = \exp(\alpha(e^s - 1))$

- Discrete Uniform Random Variable
  $S = \{1, 2, \ldots, L\}$
  $p_k = 1/L \quad k = 1, 2, \ldots, L$
  $E(X) = (L+1)/2; \mathrm{VAR}(X) = (L^2 - 1)/12$
  $M_X(s) = (1 - \exp(s(L+1)))/(1 - \exp(s))$

Continuous Random Variables

- Uniform Random Variable
  Equally likely outcomes
  $S = [a, b]$
  $f_X(x) = 1/(b-a), \quad a \leq x \leq b$
  $E(X) = (a+b)/2; \quad \mathrm{VAR}(X) = (b-a)^2/12$
  $M_X(s) = (\exp(bs) - \exp(as))/(s(b-a))$

- Exponential Random Variable, parameter λ
  $S = [0, \infty)$
  $f_X(x) = \lambda\exp(-\lambda x), \quad x \geq 0, \ \lambda > 0$
  $E(X) = 1/\lambda; \quad \mathrm{VAR}(X) = 1/\lambda^2$
  $M_X(s) = \lambda/(\lambda - s)$

- One Gaussian Random Variable, parameters μ, σ^2
  $S = (-\infty, \infty)$
  $f_X(x) = \exp(-(x-\mu)^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$
  $E(X) = \mu; \quad \mathrm{VAR}(X) = \sigma^2$
  $M_X(s) = \exp(\mu s + s^2\sigma^2/2)$

- Two Joint Gaussian Random Variables, parameters m_X, σ_X^2 and m_Y, σ_Y^2
  $S_X = (-\infty, \infty)$, $S_Y = (-\infty, \infty)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}$$
$$\exp\left[\frac{-1}{2(1-\rho_{XY}^2)}\left(\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho_{XY}\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2\right)\right]$$

$E(X) = m_X; \quad \mathrm{VAR}(X) = \sigma_X^2$
$E(Y) = m_Y; \quad \mathrm{VAR}(Y) = \sigma_Y^2$

Series summations

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

Integration

$$\int xe^{ax}\ dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{ax}\ dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$