SOLUTIONS

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2019
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

FINAL EXAM: Thursday 10:30am-12:30pm

Fall 2019, T/Th 3:00-4:15pm
(December 12, 2019)

PURDUE UNIVERSITY

This is a closed book exam. There are a total of 11 problems.
Neither calculators nor help sheets are allowed. Some formulae and helpful information is attached at the end.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK of the pages!**.

Name: ____________________ PUID: ____________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

Problem 1. (TRUE/FALSE: 4 POINTS EACH, TOTAL 16 POINTS)

For each of the following statements, determine which is valid.
If you show your reasoning you might get partial credit.
Finding a counter-example might be helpful if the answer is FALSE.
(Note: if a statement is not always true, then it is FALSE.)

Clearly label each statement T or F in the box to the left of the problem.

F

(a) if $Z = X + Y$, then the PDF of Z is the convolution of the PDFs of X and Y.

need independence of X and Y for this to be true

T

(b) Let $X(t)$ be a wide sense stationary random process input to a linear time invariant system with impulse response $h(t)$, and let $Y(t)$ be the output of the system. Then the power-spectral density of Y is $S_Y(f) = |H(f)|^2 S_X(f)$

this was given, and proven, in class

T

(c) If $E(X) = a$, then $P(X \leq a) > 0$.

F

(d) If X is a geometric RV, then $P(X > 3|X > 2) = P(X > 3)$.

$$P(X>3 \mid X>2) = \frac{P(X>3 \cap X>2)}{P(X>2)}$$

$$= \frac{P(X>3)}{P(X>2)} = \frac{(1-p)^3}{(1-p)^2} = 1-p = P(X>1)$$

OR: Geometric is memoryless so $= P(X > (3-2))$

$= P(X>1)$

Problem 2. (Multiple choice: 5 points)
If X(t) is a wide-sense stationary (WSS) random process, which of the following conditions MUST be satisfied?
If you show your reasoning you might get partial credit.

D

Answer

(a) The mean is a constant function of time.

(b) The autocorrelation function depends only on the time difference between two samples.

(c) The joint PDF of the two RVs corresponding to any two time samples depends only on the time difference between two samples.

(d) Both (a) and (b)

(e) (a), (b) and (c)

(f) None of the above

(g) Too little information to solve.

this is necessary for strict sense stationarity, but not necessary for wide sense stationarity

Problem 3. (10 POINTS (5 POINTS EACH PART))

As Baby Yoda memes flood the Internet, two main themes emerge. The first emphasizes Baby Yoda's cuteness. The second contrasts Baby Yoda to old-age Yoda.
Suppose 3/4 of all Baby Yoda (BY) memes contain the first theme, 62% contain the second theme, and 2/3 of the BY memes that contain the first theme also contain the second theme,

(a) What fraction of BY memes contain neither of these two themes?

(b) Given that a BY meme contains the second theme, what is the probability it also contains the first theme?

(Note, you can leave your answers in fractional form.)

	T_2	T_2^c	
T_1	0.5	0.25	0.75
T_1^c	0.12	0.13	0.25
	0.62	0.38	

$P(T_2 | T_1) = 2/3$

so $P(T_1 \cap T_2) = P(T_2|T_1)P(T_1) = 1/2$

a) $P(T_1^c \cap T_2^c) = 0.13$

b) $P(T_1 | T_2) = \dfrac{P(T_2 \cap T_1)}{P(T_2)} = \dfrac{0.5}{0.62}$

(a):	0.13
(b):	$\frac{50}{62} = \frac{25}{31}$

Problem 4. (10 POINTS (5 POINTS EACH PART))
Like a typical ECE student (or professor), Arthur tends to walk around campus staring at his cell phone and not looking where he's walking. As a result, he is prone to walk into something.

Suppose that X, the time that Arthur walks before he walks into something, is well modeled by an exponential random variable with mean 1. (Note: the time-units are arbitrary here, so just use the numerical value.)

$$f_X(x) = e^{-x} \quad x > 0$$

(a) What is the probability Arthur manages not to walk into something within the first 2 time units?

(b) If Arthur manages not to walk into something within the first 2 time units, what is the expected value of the time at which Arthur will walk into something?

Note: you *must* show your work to get full credit, and you may leave your answer in terms of Euler's number (that is, in terms of e).

a) $$P(X > 2) = \int_2^\infty e^{-x}\,dx = \left.\frac{e^{-x}}{-1}\right|_2^\infty$$

$$= e^{-2}$$

b) $$E(X \mid X > 2) = 2 + E(X)$$

(because memoryless)

$$= 3$$

(a):	e^{-2}
(b):	3

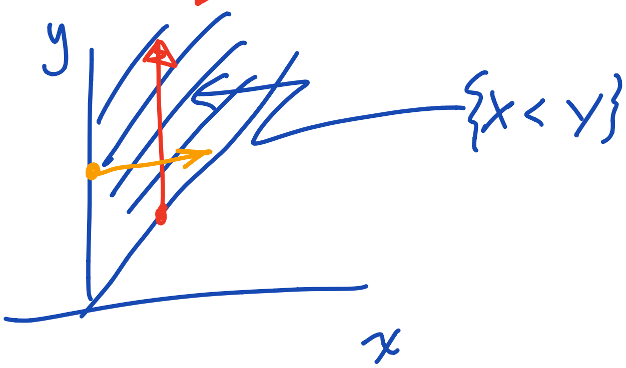
$\boxed{\text{independent}}$ ⟸ missing from problem statement

Problem 5. (5 POINTS)
Arthur has a friend Kavitha who also likes to look at her phone while walking. Suppose Y is the time that Kavitha walks before she walks into something, and this is well modeled by an exponential random variable with mean 2 (using the same time units as the previous problem.) Recall that X, the time that Arthur walks before he walks into something, is well modeled by an exponential random variable with mean 1.

What is the probability that Arthur walks into something before Kavitha walks into something?

$$\boxed{P(X<Y)} = \int_0^\infty \int_x^\infty e^{-x} \tfrac{1}{2} e^{-y/2}\, dy\, dx$$

integration option 1



$$= \int_0^\infty \int_0^y e^{-x} \frac{1}{2} e^{-y/2}\, dx\, dy$$

option 2

Option 1:

$$= \int_0^\infty e^{-x} \frac{1}{2} \left. \frac{e^{-y/2}}{-1/2} \right|_x^\infty dx$$

$$= \int_0^\infty e^{-x} e^{-x/2}\, dx = \left. \frac{e^{-3x/2}}{-3/2} \right|_0^\infty$$

$$= 2/3$$

Answer: 2/3

$E(B) = \alpha$

$E(B^2) = Var(B) + \alpha^2$

$= \alpha + \alpha^2$

$E(A) = 0$

$E(A^2) = 1$

Problem 6. (5 POINTS)
If the random process $X(t) = At + B$, where A is a Gaussian random variable with mean 0 and variance 1, and B is a Poisson random variable with parameter α, and A and B are independent, what is $R_X(t_1, t_2)$, the autocorrelation function of $X(t)$?

$$R_X(t_1, t_2) = E(X(t_1)X(t_2))$$

$$= E((At_1 + B)(At_2 + B))$$

$$= E(A^2)t_1t_2 + \underbrace{E(AB)}_{E(A)E(B)}(t_1 + t_2) + E(B^2)$$

$$= t_1t_2 + \alpha + \alpha^2$$

Answer: $t_1t_2 + \alpha + \alpha^2$

Problem 7. (6 POINTS)
$X(t)$ is a Wide Sense Stationary Random Process, with zero mean and autocorrelation function

$$R_x(\tau) = e^{-\beta|\tau|}$$

Find the variance of the random variable $X(1) - 3X(-2)$ in terms of β.

$$E(X(1) - 3X(-2)) = E(X(1)) - 3E(X(-2)) = 0$$

$$Var(X(1) - 3X(-2)) = E((X(1) - 3X(-2))^2)$$

$$= E(X(1)^2 - 6X(1)X(-2) + 9X(-2)^2)$$

$$= E(X(1)^2) - 6E(X(1)X(-2)) + 9E(X(-2)^2)$$

$$= R_x(0) - 6R_x(3) + 9R_x(0)$$

Answer: $10 - 6e^{-3\beta}$

Problem 8. (18 POINTS (6 POINTS EACH PART))
Given the Probability Density Function

$$f_X(x) = \begin{cases} (2-x)/2 & \text{when } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the CDF of X.

(b) Find $VAR(2X+1)$.

for $0 \le x < 2$

$$F_X(x) = \int_0^x \frac{(2-t)}{2}\,dt = t - \frac{t^2}{4}\Big|_0^x = x - \frac{x^2}{4}$$

$$\text{Var}(2X+1) = 4\,\text{Var}(X)$$

$$E(X) = \int_0^2 x f_X(x)\,dx = \int_0^2 \left(x - \frac{x^2}{2}\right)dx = \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^2$$

$$= 2 - \frac{8}{6} = 4/6 = 2/3$$

$$E(X^2) = \int_0^2 \left(x^2 - \frac{x^3}{2}\right)dx = \frac{x^3}{3} - \frac{x^4}{8}\Big|_0^2 = \frac{8}{3} - 2$$

$$= 2/3$$

CDF of X:	$F_X(x) = \begin{cases} 0 & x < 0 \\ x - x^2/4 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$
$VAR(2X+1) =$	$4\left(\frac{2}{3} - \left(\frac{2}{3}\right)^2\right) = 4\left(\frac{6}{9} - \frac{4}{9}\right) = 8/9$

CONTINUATION OF PROBLEM

Given the Probability Density Function

$$f_X(x) = \begin{cases} (2-x)/2 & \text{when } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find $E(X|X<1)$.

$$P(X<1) = F_X(1) = 1 - 1/4 = 3/4$$

$$f_X(x|X<1) = \begin{cases} \left(1 - x/2\right)\frac{4}{3} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

chop + scale

$$E(X|X<1) = \frac{4}{3}\int_0^1 \left(x - \frac{x^2}{2}\right)dx$$

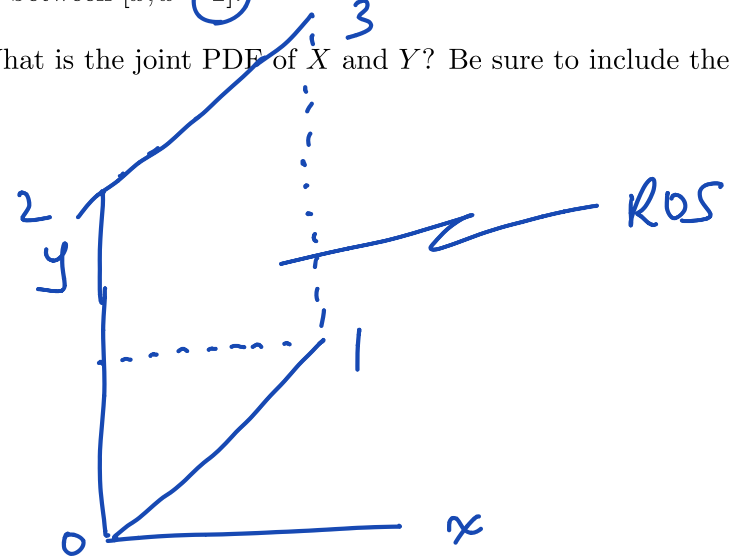
$$= \frac{4}{3}\left(\frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1\right) = \frac{4}{3}\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \frac{4}{3}\left(\frac{2}{6}\right) = \frac{4}{9}$$

$E(X|X<1) = $ 4/9

Problem 9. (10 POINTS (5 POINTS EACH PART))
Suppose X is a continuous uniform random variable between [0,1], and Y is a uniform random variable between $[x, x+2]$.

(a) What is the joint PDF of X and Y? Be sure to include the Region of Support.



$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y|x) = \begin{cases} 1/2 & x < y < 2x \\ 0 & \text{else} \end{cases}$$

(a): $f_{XY}(x,y) = \begin{cases} 1/2 & 0 \le x \le 1, \ x \le y \le 2x \\ 0 & \text{else} \end{cases}$

(b) What is $E(Y)$? (You can use iterated expectations.)

$$E(Y) = E(E(Y|X))$$
$$= E(X+1) = E(X) + 1$$
$$= 3/2$$

(b): $3/2$

Problem 10. (10 POINTS (5 POINTS EACH PART))
Given the Joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (x+y) & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_X(x|y)$, the conditional PDF of X given $Y = y$.

(b) What is $P(\{X > 1/2\} \cup \{Y > 1/2\})$? (Hint: you may find it helpful to draw a picture.)

(a)

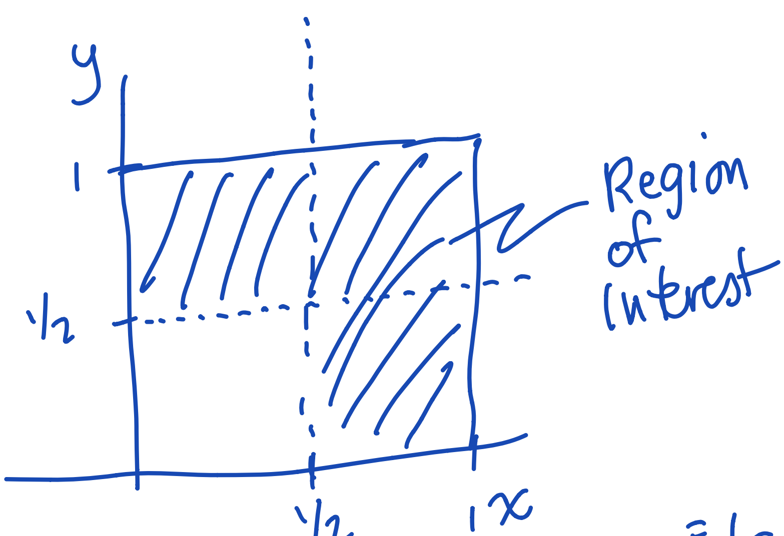
$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2} + xy\right|_0^1$$

$$= \frac{1}{2} + y \quad \text{when} \quad 0 \le x \le 1$$

$$f_X(x|y) = \begin{cases} \dfrac{x+y}{1/2+y} & \text{when } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

(b)



$P(\text{event}) =$

$= 1 - P(X < 1/2 \cap Y < 1/2)$

$$= 1 - \int_0^{1/2}\int_0^{1/2} (x+y)\,dx\,dy$$

$$= 1 - \int_0^{1/2} \left(\frac{x^2}{2} + xy\right|_0^{1/2} dy$$

$$= 1 - \int_0^{1/2} \left(\frac{1}{8} + \frac{y}{2}\right) dy$$

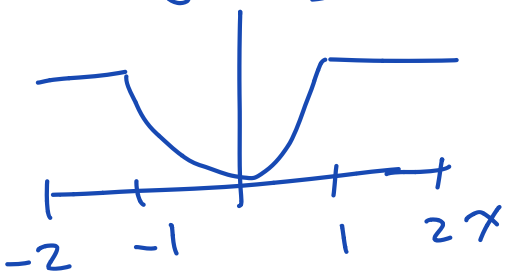
$$= 1 - \left(\frac{y}{8} + \frac{y^2}{4}\right|_0^{1/2} = 1 - \left(\frac{1}{16} + \frac{1}{16}\right) = 1 - \frac{1}{8} = 7/8$$

$$f_X(x) = \begin{cases} \frac{1}{4} & -2 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

Problem 11. (5 POINTS)

Let $Y = \min(1, X^2)$, and let X be a continuous random variable that is uniformly distributed on the interval $[-2, 2]$. What is the PDF of Y?

$g(x) = y$



2 step process

3 regions of y:

$y < 0$: $F_Y(y) = 0$

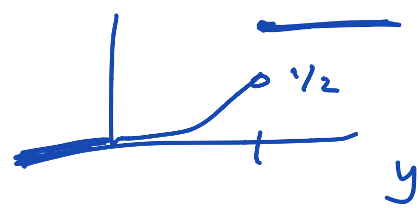
$0 \le y < 1$ → see below

$y \ge 1$ $F_Y(y) = 1$

$0 \le y < 1$

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$F_Y(y)$



$$F_X(1) = 3/4 \; ; \; F_X(-1) = 1/4$$

$F_Y(0.999) \approx 1/2$

$$f_Y(y) = \begin{cases} 0 & y < 0, y > 1 \\ 2 f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{2}\delta(y-1) & y = 1 \end{cases} = \begin{cases} \frac{1}{4\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{2}\delta(y-1) & y = 1 \\ 0 & \text{else} \end{cases}$$

check:

$$\int_0^1 \frac{1}{4\sqrt{y}}\,dy = \frac{1}{2}\,? \quad = \int_0^1 \frac{1}{4} y^{-1/2}\,dy = \frac{1}{4} \left.\frac{y^{1/2}}{1/2}\right|_0^1$$
$$= \frac{1}{4}\,\frac{2}{1}\,(1-0) = \frac{1}{2} \text{ yup.}$$

Extra space to solve – label problem clearly so I can give you credit!

ALSO: Write at the bottom of original problem that you're solving here, so I know to look here!